Quasi stationary distributions and Fleming Viot Processes

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Example

Continuous time Markov chain in

$\Lambda = \{0, 1, 2\}$

with transition rates

$q(1,0) = q(1,2) = q(2,1) = 1.\ q(0,1) = q(0,2) = 0$

(0 is absorbing state).

If one starts with 10.000 (say) chains in state 1, which proportion of the survival chains will be in state 1 by time 1?

And by time ∞?

Example 2.1 of Burdzy, Holyst and March

$\Lambda = \{0,1,2\}$ and $q(1,0) = q(1,2) = q(2,1) = 1$.

$$\nu(1) = \frac{3-\sqrt{5}}{2} = 1 + \phi$$

and

$$\nu(2) = -\frac{1-\sqrt{5}}{2} = -\phi$$

$$\phi = \frac{1-\sqrt{5}}{2} = -0.618033989$$

(golden number)

Quasi stationary distributions (QSD)

Irreducible jump Markov process with rates $Q = (q(x, y))$ on

$\Lambda \cup \{0\}$. $P_t(x, y)$ transition matrix.

Λ countable and 0 absorbing state.

Z_t is ergodic with a unique invariant measure δ_0

Law starting with μ conditioned to non absorption until time t:

$$\varphi_t^\mu(x) = \frac{\sum_{y\in\Lambda} \mu(y) P_t(y, x)}{1 - \sum_{y\in\Lambda} \mu(y) P_t(y, 0)}, \quad x \in \Lambda.$$

A *quasi stationary distribution* (QSD) is a probability measure ν on Λ satisfying

$$\varphi_t^\nu = \nu$$

ν is a left eigenvector for the restriction of the matrix Q to Λ with eigenvalue $\lambda_\nu = -\sum_{y\in\Lambda} \nu(y)q(y,0)$: ν must satisfy the system

$$\sum_{y\in\Lambda} \nu(y)\, q(y,x) = \Big(-\sum_{y\in\Lambda} \nu(y)q(y,0)\Big)\nu(x), \quad \forall x \in \Lambda.$$

$$\nu Q = \lambda_\nu \nu$$

$$\sum_{y\in\Lambda} \nu(y)\left[q(y,x) + q(y,0)\nu(x)\right] = 0, \quad \forall x \in \Lambda.$$

$$\text{recall} \qquad q(x,x) = -\sum_{y\in\Lambda\cup\{0\}\setminus\{x\}} q(x,y)$$

$$\sum_{y\in\Lambda\setminus\{x\}} \nu(y)\left[q(y,x) + q(y,0)\nu(x)\right] = \nu(x) \sum_{y\in\Lambda\setminus\{x\}} \big(q(x,y) + q(x,0)\nu(y)\big)$$

(balance equations)

Yaglom limit for μ:

$$\lim_{t\to\infty} \varphi_t^\mu(y)\,, \quad y \in \Lambda$$

if it exists and it is a probability on Λ.

Λ finite, Darroch and Seneta (1967): there exists a unique QSD ν for Q and that the Yaglom limit converges to ν independently of the initial distribution.

Λ infinite: neither existence nor uniqueness of QSD are guaranteed.

Example: asymmetric random walk Seneta:
$p = q(i, i+1) = 1 - q(i, i-1)$, for $i \geq 0$. In this case there are infinitely many QSD when $p < 1/2$ and none when $p \geq 1/2$.

Minimal QSD (for $p < 1/2$):

$$\nu(x) \sim x\Big(\frac{p}{1-p}\Big)^{x/2}$$

Existence

For $\Lambda = \mathbb{N}$ under the condition

$$\lim_{x \to \infty} \mathbb{P}(R > t | Z_0 = x) = 1 \qquad \text{for each } t > 0$$

where R absorption time,

$$\text{existence of QSD} \iff \mathbb{E} e^{\theta R} < \infty$$

for some $\theta > 0$.

(Ferrari, Kesten, Martínez and Picco [6])

Existence

Ergodicity coefficient of Q:

$$\alpha = \alpha(Q) := \sum_{z\in\Lambda} \inf_{x\in\Lambda\setminus\{z\}} q(x,z)$$

Maximal absorbing rate of Q:

$$C = C(Q) := \sup_{x\in\Lambda} q(x,0)$$

Theorem 1. *If $\alpha > C$ then there exists a unique* QSD *ν for Q and the Yaglom limit converges to ν for any initial measure μ.*

Jacka and Roberts [10]: under $\alpha > C$ uniqueness and Yaglom limit.

The Fleming-Viot process (FV) .

• System of $N > 0$ particles evolving on Λ.

• Particles move independently with rates Q between absorptions.

• When a particle is absorbed, it chooses one of the other particles uniformly and jumps instantaneously to its position.

Generator (Master equation):

$$\mathcal{L}f(\xi) = \sum_{i=1}^{N} \sum_{y \in \Lambda \setminus \{\xi(i)\}} \Big[q(\xi(i), y) + q(\xi(i), 0)\, \frac{\eta(\xi, y)}{N-1} \Big] (f(\xi^{i,y}) - f(\xi))$$

where $\xi^{i,y}(j) = y$ for $j = i$ and $\xi^{i,y}(j) = \xi(j)$ otherwise and

$$\eta(\xi, y) := \sum_{i=1}^{N} \mathbf{1}\{\xi(i) = y\}.$$

Empirical profile and conditioned process

ξ_t process in $\Lambda^{(1,\dots,N)}$;

$\eta_t \in \{\eta \in \mathbb{N}^\Lambda : \sum_x \eta(x) = N\}$ unlabeled process,

$\eta_t(x)$ = number of ξ particles in state x at time t.

Theorem 2. *Let μ probability on Λ. Assume $(\xi_0^{N,\mu}(i),\, i = 1,\dots,N)$ iid with law μ. Then, for $t > 0$ and $x \in \Lambda$,*

$$\begin{aligned}
\lim_{N\to\infty} \frac{\mathbb{E}\eta_t^{N,\mu}(x)}{N} &= \varphi_t^\mu(x) \\
\lim_{N\to\infty} \frac{\eta_t^{N,\mu}(x)}{N} &= \varphi_t^\mu(x), \qquad \textit{in Probability}
\end{aligned}$$

Fleming and Viot [8], Burdzy, Holyst and March [1], Grigorescu and Kang [9] and Löbus [12] in a Brownian motion setting.

Ergodicity of FV

Λ finite, FV Markov in finite state space

Hence ergodic (there exists unique stationary measure and the process converges to the stationary measure).

For Λ infinite:

Theorem 3. *If $\alpha > 0$, then for each N the FV process with N particles is ergodic.*

Stationary empirical profile and QSD

Assume ergodicity.

Let η^N be distributed with the unique invariant measure.

Theorem 4. $\alpha > C$. *For each* $x \in \Lambda$, *the following limits exist*

$$\lim_{N\to\infty} \frac{\eta^N(x)}{N} \quad = \quad \nu(x), \qquad \textit{in Probability}$$

and ν *is the unique* QSD *for* Q.

Sketch of proofs

Existence part of Theorem 1 is a corollary of Theorem 4.

Uniqueness: Jacka and Robert.

Theorem 3: stationary version of the process “from the past” as in perfect simulation.

Theorems 2 and 4 based on asymptotic independence.

- φ_t unique solution of

$$\frac{d}{dt}\varphi_t^\mu(x) = \sum_{y\in\Lambda} \varphi_t^\mu(y)[q(y,x) + q(y,0)\varphi_t^\mu(x)], \qquad x \in \Lambda$$

- η_t satisfies

$$\frac{d}{dt}\mathbb{E}\Big(\frac{\eta_t^{N,\mu}(x)}{N}\Big) = \sum_{y\in\Lambda} \mathbb{E}\Big(\frac{\eta_t^{N,\mu}(y)}{N}\Big(q(y,x) + q(y,0)\frac{\eta_t^{N,\mu}(x)}{N-1}\Big)\Big)$$

- We prove:

$$\mathbb{E}[\eta_t^{N,\mu}(y)\,\eta_t^{N,\mu}(x)] - \mathbb{E}\eta_t^{N,\mu}(y)\,\mathbb{E}\eta_t^{N,\mu}(x) \;=\; O(N)$$

- QSD satisfies

$$\sum_{y\in\Lambda} \nu(y)\,[q(y,x)+q(y,0)\nu(x)] \;=\; 0, \quad x\in\Lambda.$$

- η^N invariant for FV satisfies:

$$\sum_{y\in\Lambda} \mathbb{E}\Big(\frac{\eta^N(y)}{N}\Big(q(y,x)+q(y,0)\frac{\eta^N(x)}{N-1}\Big)\Big) \;=\; 0, \quad x\in\Lambda.$$

- Under $\alpha > C$:

$$\mathbb{E}[\eta^N(y)\,\eta^N(x)] - \mathbb{E}\eta^N(y)\,\mathbb{E}\eta^N(x) = O(N)$$

- Variance order $1/N$, setting $x=y$.
- Finally we show $(\varphi_t^{N,\mu},\, N\in\mathbb{N})$ and $(\rho^N,\, N\in\mathbb{N})$ are tight.

Comments

• Fleming-Viot permits to show existence of a QSD in the $\alpha > C$ case (new).

• Compared with Brownian motion in a bounded region with absorbing boundary (Burdzy, Holyst and March [1], Grigorescu and Kang [9] and Löbus [12] and other related works):

• Existence of the FV process immediate here.

• they prove the convergence without factorization.

• We prove: vanishing correlations sufficient for convergence of expectations and in probability.

• To prove tightness classify ξ particles in types.

• Tightness proof needs $\alpha > C$ as the vanishing correlations proof.

Graphical construction of FV process

Graphical construction of FV process

To each particle $i = 1, \ldots, N$, associate 3 marked Poisson processes:

- Regeneration times. PP (α): $(a_n^i)_{n\in\mathbb{Z}}$, marks $(A_n^i)_{n\in\mathbb{Z}}$
- Internal times. PP $(\bar{q} - \alpha)$: $(b_n^i)_{n\in\mathbb{Z}}$, marks $((B_n^i(x),\ x \in \Lambda),\ n \in \mathbb{Z})$
- Voter times. PP (C): $(c_n^i)_{n\in\mathbb{Z}}$, marks $((C_n^i,\ (F_n^i(x),\ x \in \Lambda)),\ n \in \mathbb{Z})$

Law of marks:

- $\mathbb{P}(A_n^i = y) = \alpha(y)/\alpha$, $y \in \Lambda$;
- $\mathbb{P}(B_n^i(x) = y) = \dfrac{q(x,y) - \alpha(y)}{\bar{q} - \alpha}$, $x \in \Lambda$, $y \in \Lambda \setminus \{x\}$;
  $\mathbb{P}(B_n^i(x) = x) = 1 - \sum_{y \in \Lambda \setminus \{x\}} \mathbb{P}(B_n^i(x) = y)$.
- $P(F_n^i(x) = 1) = \dfrac{q(x,0)}{C} = 1 - P(F_n^i(x) = 0)$, $x \in \Lambda$.
- $P(C_n^i = j) = \dfrac{1}{N-1}$, $j \neq i$.

Call ω a realization of the marked PP.

Construction of $\xi^{N,\xi}_{[s,t]} = \xi^{N,\xi}_{[s,t],\omega}$

- Order Poisson times.

- Initial configuration ξ at time s .

- Configuration does not change between Poisson events.

- At each regeneration time a^i_n particle i adopts state A^i_n regardless the current configuration.

- If at the internal time b^i_n- the state of particle i is x, then at time b^i_n particle i adopts state $B^i_n(x)$ regardless the state of the other particles.

- If at the voter time c^i_n- the state of particle i is x and $F^i_n(x) = 1$, then at time c^i_n particle i adopts the state of particle C^i_n; if $F^i_n(x) = 0$, then particle i does not change state.

- The final configuration is $\xi^{N,\xi}_{[s,t]}$.

Lemma 1. *The process* $(\xi^{N,\xi}_{[s,t]}, t \geq s)$ *is* FV *with initial condition* $\xi^{N,\xi}_{[s,s]} = \xi$.

Generalized duality Define

$$\omega^i[s,t] = \{m \in \omega : m \text{ involved in the definition of } \xi^{N,\xi}_{[s,t],\omega}(i)\},$$

Generalized duality equation:

$$\xi^{N,\xi}_{[s,t],\omega}(i) = H(\omega^i[s,t], \xi). \tag{1}$$

- No explicit formula for H.
- For any time s, $\xi^{N,\xi}_{[s,t]}(i)$ depends only on the *finite* number of Poisson events contained in $\omega^i[s,t]$.

Theorem 3. *If $\alpha > 0$ the* FV *process is ergodic.*

Proof If number of marks in $\omega^i[-\infty, t]$ is finite, then

$$\xi^N_{t,\omega}(i) =: \lim_{s \to -\infty} H(\omega^i[s,t], \xi), \quad i \in \{1, \dots, N\}, \ \ t \in \mathbb{R}$$

is well defined and does not depend on ξ.

• By construction $(\xi^N_t, \, t \in \mathbb{R})$ is a stationary FV process.

• The law of ξ^N_t is unique invariant measure.

• Number of points in $\omega^i[-\infty, t]$ is finite if there is $[s(\omega), s(\omega)+1]$ in the past of t with one regeneration mark for each k and no voter marks. □

Particle correlations in the FV process

Proposition 1. *Let* $x, y \in \Lambda$*. For all* $t > 0$

$$\left|\mathbb{E}\Big(\frac{\eta_t^N(x)\eta_t^N(y)}{N^2}\Big) - \mathbb{E}\Big(\frac{\eta_t^N(x)}{N}\Big)\,\mathbb{E}\Big(\frac{\eta_t^N(y)}{N}\Big)\right| < \frac{1}{N}\, e^{2Ct} \tag{2}$$

Assume $\alpha > C$*. Let* η^N *be distributed according to the unique invariant measure for the* FV *process with* N *particles. Then*

$$\left|\mathbb{E}\Big(\frac{\eta^N(x)\eta^N(y)}{N^2}\Big) - \mathbb{E}\Big(\frac{\eta^N(x)}{N}\Big)\,\mathbb{E}\Big(\frac{\eta^N(y)}{N}\Big)\right| < \frac{1}{N}\,\frac{\alpha}{\alpha - C} \tag{3}$$

Coupling

• 4-fold coupling $(\omega^i[s,t], \omega^j[s,t], \hat{\omega}^i[s,t], \hat{\omega}^j[s,t])$

• $\omega^i[s,t] = \hat{\omega}^i[s,t]$

• $\hat{\omega}^j[s,t] \cap \omega^i[s,t] = \emptyset$ implies $\omega^j[s,t] = \hat{\omega}^j[s,t]$

• marginal process $(\hat{\omega}^i[s,t], \hat{\omega}^j[s,t])$ have the same law as two independent processes with the same marginals as $(\omega^i[s,t], \omega^j[s,t])$.

$$\mathbb{P}(\xi_t^{N,\xi}(j) = x, \xi_t^{N,\xi}(i) = y) - \mathbb{P}(\xi_t^{N,\xi}(j) = x)\mathbb{P}(\xi_t^{N,\xi}(i) = y)$$

$$= \mathbb{E}\Big(\mathbf{1}\{H(\omega^j, \xi) = x,\ H(\omega^i, \xi) = y)\} - \mathbf{1}\{H(\hat{\omega}^j, \xi) = x),\ H(\hat{\omega}^i, \xi) = y)\}\Big)$$

- If

$$\omega^i \cap \omega^j = \emptyset$$

then

$$\omega^j(s,t) = \hat{\omega}^j(s,t) \text{ and } \omega^i(s,t) = \hat{\omega}^i(s,t)$$

Hence,

$$|\mathbb{P}(\xi_t^{N,\xi}(j) = x, \xi_t^{N,\xi}(i) = y) - \mathbb{P}(\xi_t^{N,\xi}(j) = x)\mathbb{P}(\xi_t^{N,\xi}(i) = y)|$$

$$\leq \ \mathbb{P}(\omega^i \cap \omega^j \neq \emptyset).$$

Lemma 2. *($s = 0$)*

$$\mathbb{P}(\omega^i \cap \omega^j \neq \emptyset) \quad \leq \quad \frac{1}{N-1} \, \frac{C}{\alpha - C} \, (1 - e^{2(C-\alpha)t}) \tag{4}$$

Proof:

$$\mathbb{P}(\omega^i \cap \omega^j \neq \emptyset) \; \leq \; \frac{2C}{N-1} \int_0^t \mathbb{E}\hat{\Psi}^i[s,t] \, \mathbb{E}\hat{\Psi}^j[s,t] ds$$

$\hat{\Psi}^i[s,t]$ Random walk that grows with rate Cx and decreases with rate αx. Expectation is bounded by $e^{(t-s)(C-\alpha)}$.

$$\mathbb{P}(\omega^i \cap \omega^j \neq \emptyset) \leq \frac{2C}{N-1} \int_0^t e^{2(C-\alpha)s} ds$$

which gives the result. □

Proof of Proposition 1 Take η and ξ such that $\eta(x) = \sum_j \mathbf{1}\{\xi(j) = x\}$. Then

$$\mathbb{E}\Big(\frac{\eta_t^{N,\eta}(x)\eta_t^{N,\eta}(y)}{N^2}\Big) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbb{P}(\xi_t^{N,\xi}(i) = x, \xi_t^{N,\xi}(j) = y)$$

$$\frac{\mathbb{E}\eta_t^{N,\eta}(x)\,\mathbb{E}\eta_t^{N,\eta}(y)}{N^2} = \frac{1}{N^2}\Big(\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbb{P}(\xi_t^{N,\xi}(i) = x)\mathbb{P}(\xi_t^{N,\xi}(j) = y)\Big)$$

Using this, and (4) with $\alpha = 0$ we get (2).

Assume $\alpha > C$. Taking $t = \infty$ in (4) we get (3). $\square$

Tightness

Proposition 2. *For all* $t > 0$, $x \in \Lambda$, $i = 1, \ldots, N$ *and* μ,

$$\frac{\mathbb{E}\eta_t^{N,\mu}(x)}{N} \quad \leq \quad e^{Ct} \sum_{z \in \Lambda} \mu(z) P_t(z, x).$$

As a consequence $(\mathbb{E}\eta_t^{N,\mu}/N,\ N \in \mathbb{N})$ *is tight.*

Assume $\alpha > 0$ and define μ_α on Λ by

$$\mu_\alpha(x) = \frac{\alpha_x}{\alpha}, \quad x \in \Lambda,$$

where $\alpha_x = \inf_z q(z,x)$. For $z, x \in \Lambda$ define

$$R_\lambda(z,x) = \int_0^\infty \lambda e^{-\lambda t} P_t(z,x) dt.$$

Proposition 3. *Assume $\alpha > C$ and let η^N distributed with invariant measure for* FV. *Then for $x \in \Lambda$,*

$$\rho^N(x) \ \leq \ \frac{C}{\alpha - C} \, \mu_\alpha R_{(\alpha - C)}(x)$$

As a consequence, the family of measures $(\eta^N/N,\, N \in \mathbb{N})$ is tight.

Types

- Particle i is *type* 0 at time t if it has not been absorbed in the time interval $[0, t]$.

- If at absorption time s particle i jumps over particle j which has type k, then at time s particle i changes its type to $k + 1$.

$$\mathbb{P}(\xi_t^{N,\mu}(i) = x, \text{type}(i, t) = 0) = \sum_{z \in \Lambda} \mu(z) P_t(z, x).$$

$$A_t(x, k) =: \mathbb{P}(\xi_t^{N,\mu}(i) = x, \text{type}(i, t) = k)$$

Proof of Proposition 2 Recursive hypothesis:

$$A_t(x,k) \le \frac{(Ct)^k}{k!} \sum_{z\in\Lambda} \mu(z) P_t(z,x) \tag{5}$$

By (5) the statement is true for $k = 0$.

$$A_t(x,k+1) \quad \le \quad \int_0^t C \sum_{y\in\Lambda} A_s(y,k)\, P_{t-s}(y,x)\, ds.$$

Using recursive hypothesis,

$$= \quad \int_0^t C\, \frac{(Cs)^k}{k!} \sum_{z\in\Lambda} \mu(z) \sum_{y\in\Lambda} P_s(z,y) P_{t-s}(y,x) ds$$

$$= \quad \frac{(Ct)^{k+1}}{(k+1)!} \sum_{z\in\Lambda} \mu(z) P_t(z,x).$$

by Chapman-Kolmogorov. This proves (5). □

Proof of Proposition 3 Under the hypothesis $\alpha > C$ the process

$$((\xi_t^N(i), \mathrm{type}(i,t)), i = 1, \ldots, N), \, t \in \mathbb{R})$$

is Markovian constructed in a stationary way

$$A(x,k) := \mathbb{P}(\xi_s^N(i) = x, \mathrm{type}(i,s) = k)$$

does not depend on s.

Last regeneration mark of site i before time s happened at time $s - T_\alpha^i$, where T_α^i is exponential of rate α. Then,

$$A(x,0) \;=\; \int_0^\infty \alpha e^{-\alpha t} \sum_{z \in \Lambda} \mu_\alpha(z) P_t(z,x) dt \;=\; \mu_\alpha R_\alpha(x).$$

Similar reasoning implies

$$
\begin{aligned}
A(x,k) &\leq \int_0^\infty e^{-\alpha t} C \sum_{z\in\Lambda} A(z,k-1) P_s(z,x)\, dt. \\
&= \frac{C}{\alpha} A_{k-1} R_\alpha(x) \;\leq\; \Big(\frac{C}{\alpha}\Big)^k \mu_\alpha R_\alpha^{k+1}(x).
\end{aligned}
$$

$R_\lambda^k(z,x)$ expectation of $P_{\tau_k}(z,x)$, τ_k sum of k independent exponential λ. Multiplying and dividing by $(\alpha - C)$,

$$
\mathbb{P}(\xi_s^N(i) = x) \;\leq\; \frac{C}{\alpha - C} \sum_{k=0}^\infty \Big(\frac{C}{\alpha}\Big)^k \Big(1 - \frac{C}{\alpha}\Big) \mu_\alpha R_\alpha^{k+1}(x)
$$

Expectation of $\mu_\alpha R_\alpha^K$, K geometric with $p = 1 - (C/\alpha)$.

$$
\mathbb{P}(\xi_s^N(i) = x) \;\leq\; \frac{C}{\alpha - C}\, \mu_\alpha R_{\alpha - C}(x). \qquad \Box
$$

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