A New Keynesian Model with Heterogeneous Expectations

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Preliminary
Motivation
Motivation

- Extend Branch and McGough (2004) to an applied model with micro-foundations
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- Introduce heterogeneity and bounded rationality at the agent level
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- Extend Branch and McGough (2004) to an applied model with micro-foundations
- Introduce heterogeneity and bounded rationality at the agent level
- Maintain tractability and comparability
Motivation

- Extend Branch and McGough (2004) to an applied model with micro-foundations
- Introduce heterogeneity and bounded rationality at the agent level
- Understand what is needed to maintain tractability and comparability
Motivation

Why heterogeneous bounded rationality?
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Why heterogeneous bounded rationality?

- Researchers’ prerogative
Motivation

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- Potential for complex dynamics (Brock and Hommes (1997), Sethi and Frank (1995))
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Why heterogeneous bounded rationality?

- Researchers’ prerogative
- Potential for complex dynamics (Brock and Hommes (1997), Sethi and Frank (1995))
- Empirical evidence supports HBR (Mankiw, Reis, Wolfers (2003), Carrol (2003), Branch (2004))
Motivation

Why analyze the New Keynesian model?
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- Familiar construction, but one which relies on homogeneity of expectations
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- Familiar construction, but one which relies on homogeneity of expectations
- Expectations are known to play an important role in model’s dynamics
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- Familiar construction, but one which relies on homogeneity of expectations
- Expectations are known to play an important role in model’s dynamics
- Applied interest in its determinacy properties
Motivation

Issues
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- Discipline bounded rationality
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- “Optimal” behavior in a boundedly rational environment
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- “Optimal” behavior in a boundedly rational environment
- Complete markets and Calvo risk
Motivation

Issues

- Discipline bounded rationality
- “Optimal” behavior in a boundedly rational environment
- Complete markets and Calvo risk
- What about bonds?
Boundedly rational expectations

Goal: Identify characteristics appropriate for an expectations operator.
Boundedly rational expectations

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Assume two types of agents.
Goal: Identify characteristics appropriate for an expectations operator.

Assume two types of agents.

Let $E_t^\tau(x_{t+k})$ be the time $t$ expectation of $x_{t+k}$ as formed by an agent of type $\tau$. 
Boundedly rational expectations

Axioms for expectations
Boundedly rational expectations

Axioms for expectations

- Fixes observables
Boundedly rational expectations

Axioms for expectations

- Fixes observables
- If $x$ is a variable forecasted by agents and has steady state $\bar{x}$ then $E^1\bar{x} = E^2\bar{x} = \bar{x}$. 
Boundedly rational expectations

Axioms for expectations

- Fixes observables
- If $x$ is a variable forecasted by agents and has steady state $\bar{x}$ then $E^1 x = E^2 x = \bar{x}$.
- If $x, y, x+y$ and $\alpha x$ are variables forecasted by agents then $E_t^\tau (x+y) = E_t^\tau (x) + E_t^\tau (y)$ and $E_t^\tau (\alpha x) = \alpha E_t^\tau (x)$. 
Boundedly rational expectations

Axioms for expectations

- Fixes observables
- If $x$ is a variable forecasted by agents and has steady state $\bar{x}$ then $E^1 \bar{x} = E^2 \bar{x} = \bar{x}$.
- If $x, y, x + y$ and $\alpha x$ are variables forecasted by agents then $E^\tau_t (x + y) = E^\tau_t (x) + E^\tau_t (y)$ and $E^\tau_t (\alpha x) = \alpha E^\tau_t (x)$.
- If for all $k$, $x_{t+k}$ and $\sum_k \beta^{t+k} x_{t+k}$ are forecasted by agents then

$$E^\tau_t \left( \sum_k \beta^{t+k} x_{t+k} \right) = \sum_k \beta^{t+k} E^\tau_t (x_{t+k}).$$
Boundedly rational expectations

Axioms for expectations
Boundedly rational expectations

Axioms for expectations

$E_t^\tau$ satisfies the law of iterated expectations (L.I.E.): If $x$ is a variable forecasted by agents at time $t$ and time $t + k$ then $E_t^\tau \circ E_{t+k}^\tau(x) = E_t^\tau(x)$. 
Boundedly rational expectations

Axioms for expectations

- $E_t^\tau$ satisfies the law of iterated expectations (L.I.E.): If $x$ is a variable forecasted by agents at time $t$ and time $t + k$ then
  \[ E_t^\tau \circ E_{t+k}^\tau(x) = E_t^\tau(x). \]
- If $x$ is a variable forecasted by agents at time $t$ and time $t + k$ then
  \[ E_t^\tau E_{t+k}^{\tau'}(x) = E_t^\tau x, \quad \tau' \neq \tau. \]
Boundedly rational expectations

Axioms for expectations

- $E_t^\tau$ satisfies the law of iterated expectations (L.I.E.): If $x$ is a variable forecasted by agents at time $t$ and time $t + k$ then $E_t^\tau \circ E_{t+k}^\tau(x) = E_t^\tau(x)$.
- If $x$ is a variable forecasted by agents at time $t$ and time $t + k$ then $E_t^\tau E_{t+k}^\tau(x) = E_t^\tau x$, $\tau' \neq \tau$.
- All agents have common expectations on expected differences in limiting income.
Yeoman Farmers
Yeoman Farmers

Agent’s behavior
Yeoman Farmers

Agent’s behavior

- Agents indexed by $[0, 1], \ i \in [0, \alpha] \implies \text{agent } i \text{ is of type 1.}$
Yeoman Farmers

Agent’s behavior

- Agents indexed by $[0, 1]$, $i \in [0, \alpha] \implies$ agent $i$ is of type 1.
- Utility: $u(C^i) - v(N^i)$
Agent’s behavior

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Agent’s behavior

- Agents indexed by $[0, 1]$, $i \in [0, \alpha] \implies$ agent $i$ is of type 1.
- Utility: $u(C^i) - v(N^i)$
- Current variables may not be observable but choice variables are.
- Agents choices must satisfy their perceived FOC.
An IS relation
An IS relation

Consumption and bonds
An IS relation

Consumption and bonds

- Consumption Euler Equation:

\[ u_c(C^i_t) = \beta (1 + i_t) E_t^\tau \left( \frac{P_t}{P_{t+1}} \right) u_c(C^i_{t+1}) . \]
An IS relation

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- Real Income: \( \Omega^\tau_t \) = average real income across agents of type \( \tau \).
An IS relation

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- Real Income: \( \Omega^\tau_t = \) average real income across agents of type \( \tau \).

- Equilibrium insurance: \( I^i_{rt} = P_t \Omega^\tau_t \) and \( I^i_{pt} = P_t Y^i_t \).
An IS relation

Consumption and bonds

- **Consumption Euler Equation:**

  \[
  u_c(C_t^i) = \beta (1 + i_t) E_t^\tau \left( \frac{P_t}{P_{t+1}} \right) u_c(C_{t+1}^i).
  \]

- **Real Income:** \( \Omega_t^{\tau} = \) average real income across agents of type \( \tau \).

- **Equilibrium insurance:** \( I_{rt}^i = P_t \Omega_t^{\tau} \) and \( I_{pt}^i = P_t^i Y_t^i \).

- **Log-linearized equilibrium consumption:**

  \[
  c_t^i = \hat{\Omega}_t^{\tau} \equiv \omega_t^{\tau} + \beta^{-1} \frac{b_{t-1}^i}{\bar{Y}} - \frac{b_t^i}{\bar{Y}},
  \]
An IS relation

Consumption and bonds
Log-linear equilibrium Euler stepped forward: Let \( \hat{\Omega}_\infty = \lim_{k \to \infty} E^\tau_t \hat{\Omega}^\tau_{t+k} \). Then

\[
\hat{\Omega}^\tau_t = \hat{\Omega}^\tau_\infty - \sigma^{-1} E^\tau_t \sum_{k \geq 0} (i_{t+k} - \pi_{t+k+1}) ,
\]
An IS relation

Consumption and bonds

- Log-linear equilibrium Euler stepped forward: Let 
  \( \hat{\Omega}_\infty = \lim_{k \to \infty} E_t^\tau \hat{\Omega}_{t+k}^\tau \). Then
  \[
  \hat{\Omega}_t^\tau = \hat{\Omega}_\infty^\tau - \sigma^{-1} E_t^\tau \sum_{k \geq 0} (i_{t+k} - \pi_{t+k+1}),
  \]

- Aggregate: Let \( \hat{E} = \alpha E^1 + (1 - \alpha) E^2 \). Then
  \[
  y_t = \hat{E}_t y_{t+1} - \sigma^{-1} (i_t - \hat{E}_t \pi_{t+1}) + \alpha \hat{\Omega}_\infty^1 + (1 - \alpha) \hat{\Omega}_\infty^2 - \hat{E}_t \left( \alpha \hat{\Omega}_\infty^1 + (1 - \alpha) \hat{\Omega}_\infty^2 \right),
  \]
An IS relation

Axiom (A6) provides

\[ E_t^\tau \left( \hat{\Omega}^{\tau'}_{\infty} \right) = \hat{\Omega}^{\tau'}_{\infty}, \]

which implies

\[ \alpha \hat{\Omega}^1_{\infty} + (1 - \alpha) \hat{\Omega}^2_{\infty} - E_t \left( \alpha \hat{\Omega}^1_{\infty} + (1 - \alpha) \hat{\Omega}^2_{\infty} \right) = 0. \]
Proposition

If agents’ expectations $E^1$ and $E^2$ satisfy assumptions (A1)-(A6) then, up to a log-linear approximation, equilibrium output and inflation satisfy the following IS relation

$$y_t = \hat{E}_t y_{t+1} - \sigma^{-1} (i_t - \hat{E}_t \pi_{t+1})$$

(1)

where $\hat{E} = \alpha E^1 + (1 - \alpha) E^2$. 
An AS relation
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Setting prices
An AS relation

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▶ Demand for individual goods is as usual.
An AS relation

Setting prices

- Demand for individual goods is as usual.
- Dismiss free rider problem.
An AS relation

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- Let $C^i_{t+k} = C^i_{t+k}(P^i_t)$ be the consumption bundle in the event that agent $i$ cannot change prices for $k$ periods.
An AS relation

Setting prices

- Demand for individual goods is as usual.
- Dismiss free rider problem.
- Let $C_{i,t+k}^i = C_{i,t+k}^i(P_t^i)$ be the consumption bundle in the event that agent $i$ can not change prices for $k$ periods.
- $P_t^i$ is chosen by contract to solve

$$\max E_t^\tau \sum_{k \geq 0} (\beta \gamma)^k \left[ u(C_{t+k}^i(P_t^i), \cdot) - v \left( \left( \frac{P_t^i}{P_{t+k}} \right)^{-\theta} Y_{t+k} \right) \right].$$
An AS relation

Setting prices

- Log-linear FOC and impose equilibrium consumption:

\[ E_t^\tau \sum_{k \geq 0} (\gamma \beta)^k (\log (P_i^t) - \log (P_{t+k}) - \zeta_1 \hat{\Omega}_{t+k} - \zeta_2 y_{t+k}) = 0. \]
An AS relation

Setting prices

- Log-linear FOC and impose equilibrium consumption:

\[ E_t^\tau \sum_{k \geq 0} (\gamma \beta)^k \left( \log (P_t^i) - \log (P_{t+k}) - \zeta_1 \hat{\Omega}^\tau_{t+k} - \zeta_2 y_{t+k} \right) = 0. \]

- Proceed as usual to get
An AS relation

Proposition

If agents’ expectations $E^1$ and $E^2$ satisfy assumptions (A1)-(A6) then, up to a log-linear approximation, equilibrium output and inflation satisfies the AS relation

$$\pi_t = \beta \hat{E}_t \pi_{t+1} + \lambda_1 y_t + \lambda_2 \hat{E}_t y_t + \frac{1 - \gamma}{\gamma} (\hat{E}_t \pi_t - \pi_t). \quad (2)$$

where $\hat{E} = \alpha E^1 + (1 - \alpha) E^2$. 
An Application to Monetary Policy

(Nobody likes our application but we don’t care)
An Application to Monetary Policy

Model:

\[ y_t = \hat{E}_t y_{t+1} - \sigma^{-1} (i_t - \hat{E}_t \pi_{t+1}) \]

\[ \pi_t = \beta \hat{E}_t \pi_{t+1} + \lambda_1 y_t + \lambda_2 \hat{E}_t y_t + \frac{1 - \gamma}{\gamma} (\hat{E}_t \pi_t - \pi_t) \]
An Application to Monetary Policy

Model:

\[ y_t = \hat{E}_t y_{t+1} - \sigma^{-1}\left( i_t - \hat{E}_t \pi_{t+1} \right) \]

\[ \pi_t = \beta \hat{E}_t \pi_{t+1} + \lambda_1 y_t + \lambda_2 \hat{E}_t y_t + \frac{1 - \gamma}{\gamma} \left( \hat{E}_t \pi_t - \pi_t \right) \]

Monetary Policy:

PR\(_1\) : \[ i_t = \alpha_y \hat{E}_t y_{t+1} + \alpha_\pi \hat{E}_t \pi_{t+1}, \]
An Application to Monetary Policy

Model:

\[ y_t = \hat{E}_t y_{t+1} - \sigma^{-1} (i_t - \hat{E}_t \pi_{t+1}) \]
\[ \pi_t = \beta \hat{E}_t \pi_{t+1} + \lambda_1 y_t + \lambda_2 \hat{E}_t y_t + \frac{1 - \gamma}{\gamma} (\hat{E}_t \pi_t - \pi_t) \]

Monetary Policy:

\[ PR_1 : i_t = \alpha_y E_t y_{t+1} + \alpha_\pi E_t \pi_{t+1}, \]
\[ PR_2 : i_t = \alpha_y \hat{E}_t y_{t+1} + \alpha_\pi \hat{E}_t \pi_{t+1}. \]
Expectations
Expectations

Type 1: “Perfect Foresight”
Expectations

Type 1: “Perfect Foresight”

Type 2: $E_t^2 x_t = \theta_x x_{t-1}$
Expectations

Type 1: “Perfect Foresight”

Type 2: $E_t^2 x_t = \theta_x x_{t-1}$

Aggregate:

$$\hat{E}_t x_t = n_x x_t + (1 - n_x) \theta_x x_{t-1}$$
$$\hat{E}_t(x_{t+1}) = n_x x_{t+1} + (1 - n_x) \theta_x^2 x_{t-1}.$$
Model
Full Model

\[
y_t = \hat{E}_t y_{t+1} - \sigma^{-1} (i_t - \hat{E}_t \pi_{t+1}) \quad (3)
\]

\[
\pi_t = \beta \hat{E}_t \pi_{t+1} + \lambda_1 y_t + \lambda_2 \hat{E}_t y_t + \frac{1 - \gamma}{\gamma} (\hat{E}_t \pi_t - \pi_t) \quad (4)
\]

\[
i_t = \alpha_y E_t y_{t+1} + \alpha_\pi E_t \pi_{t+1} \quad (5)
\]

\[
\hat{E}_t y_t = n_y y_t + (1 - n_y) \theta_y y_{t-1}
\]

\[
\hat{E}_t (y_{t+1}) = n_y E_t y_{t+1} + (1 - n_y) \theta_y^2 y_{t-1}
\]

\[
\hat{E}_t \pi_t = n_\pi \pi_t + (1 - n_\pi) \theta_\pi \pi_{t-1}
\]

\[
\hat{E}_t (\pi_{t+1}) = n_\pi E_t \pi_{t+1} + (1 - n_\pi) \theta_\pi^2 \pi_{t-1},
\]
Determinacy
Determinacy

Definition
The Model is Determinate if there is a unique bounded solution, indeterminate if there are multiple bounded solutions and explosive otherwise.
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Stacked Version

\[ z_t = M^{-1} E_t z_{t+1} \]  \hspace{1cm} (6)

where \( x = (y, \pi)' \) and \( z = (x', x'_{-1})' \).
Determinacy

Definition
The Model is Determinate if there is a unique bounded solution, indeterminate if there are multiple bounded solutions and explosive otherwise.

Stacked Version

\[ z_t = M^{-1} E_t z_{t+1} \]  
(6)

where \( x = (y, \pi)' \) and \( z = (x', x'_{-1})' \).

- Determinate if and only if precisely two eigenvalues of \( M^{-1} \) are outside the unit circle.
- Order \( k \) indeterminate if \( k + 2 \) eigenvalues are outside the unit circle.
- else, explosive.
$\theta = 0.9$, $\gamma = 0.65$, $\phi = 6.3694$, $\lambda = 0.024$
\[ \theta = 1.1, \gamma = 0.65, \phi = 6.3694, \lambda = 0.024 \]
Indeterminacy and Dynamic Stability

Recall the stacked model, \( z_t = M(n)^{-1}E_t z_{t+1} \).
Indeterminacy and Dynamic Stability

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Under perfect foresight, $z_t = M(n) z_{t-1}$. 
Indeterminacy and Dynamic Stability

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Under perfect foresight, \( z_t = M(n)z_{t-1} \).

This system is dynamically stable if and only if the stacked model is order 2 indeterminate.
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Under perfect foresight, \( z_t = M(n)z_{t-1} \).

This system is dynamically stable if and only if the stacked model is order 2 indeterminate.

Now couple it with endogenously evolving predictor proportion \( n \).
Indeterminacy and Dynamic Stability

Model: \( z_t = M(n)z_{t-1} \).

Taylor principle satisfied, determinate under rationality.
Indeterminacy and Dynamic Stability

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- For \( n = 1 \), model is determinate/unstable
Indeterminacy and Dynamic Stability

Model: $z_t = M(n)z_{t-1}$.

Taylor principle satisfied, determinate under rationality.

- For $n = 1$, model is determinate/unstable
- For large $n$ less than one, model is order 2 indeterminate/stable
Indeterminacy and Dynamic Stability

Model: \( z_t = M(n)z_{t-1} \).

Taylor principle satisfied, determinate under rationality.

- For \( n = 1 \), model is determinate/unstable
- For large \( n \) less than one, model is order 2 indeterminate/stable
- For smaller \( n \), model is determinate or order 1 indeterminate/unstable
\[ C = 0.0216 \]

\[ \theta_{\pi} = 1.1, \theta_{y\pi} = 1.0101, \alpha_{\pi} = 0.7, \alpha_{y} = 0.4, \omega = 4 \]
$\theta_\pi = 1.5, \theta_y = 1.5, \alpha_\pi = 1.4, \alpha_y = 0.35, \omega = 4$
Conclusion
Conclusion
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- With appropriate restrictions, aggregation across expectations’ types is tractable and yields IS and AS relations similar in form to the homogeneous case.
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- The presence of adaptive agents may significantly alter the model’s determinacy properties.
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- With appropriate restrictions, aggregation across expectations’ types is tractable and yields IS and AS relations similar in form to the homogeneous case.
- The presence of adaptive agents may significantly alter the model’s determinacy properties.
- Dynamic predictor selection makes pretty pictures.