A linear path towards synchronization:

Reformulating the Kuramoto model of self-synchronizing coupled oscillators

Network Synchronization:
From dynamical systems to neuroscience
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Introduction

Systems exhibiting spontaneous synchronization of coupled oscillators

Biology
- Neural firing patterns
- Clouds of fireflies flashing simultaneously
- Coordinated action of cardiac pacemaker cells

Physics
- Josephson junction arrays
- Collective atomic recoil lasing
- Flavour evolution of oscillating neutrinos
- Phase locking of oscillations in coupled chemical reactions
The Kuramoto model
Paradigm of spontaneous synchronization

System with a **heterogeneous** population of $N$ oscillators:

$$\dot{\theta}_k = \omega_k + \sum_{j \neq k}^N K_{j,k} \sin(\theta_j - \theta_k)$$

**Weak coupling**, so can ignore amplitude

**Phase order parameter:**

$$r e^{i\phi} \equiv \frac{1}{N} \sum_{j=1}^N e^{i\theta_j}$$

**Synchronization defined:**

If $r$ reaches a steady state then

$$\begin{cases} 
    r = 0 & \text{no synchronization} \\
    0 < r < 1 & \text{partial synchronization} \\
    r = 1 & \text{full synchronization}
\end{cases}$$
Synchronization defined

Incoherent.
Not phase locked
or synchronized
Synchronization defined

\[ r \approx 0 \]

Incoherent.
Not phase locked or synchronized

\[ r \approx 1 \]

Partially synchronized
Synchronization defined

If \( r \) reaches a steady state then

\[
\begin{cases}
  r = 0 & \text{no synchronization} \\
  0 < r < 1 & \text{partial synchronization} \\
  r = 1 & \text{full synchronization}
\end{cases}
\]

- **Incoherent.** Not phase locked or synchronized
- **Partially synchronized**
- **"Splay state"**: All phase locked but not synchronized

Phase locking ≠ synchronization
The Kuramoto solution

Assuming uniform global coupling, $K_{jk} = K/N$

Rotating frame so $\phi = 0$

Implicit solution for $r$:

$$r = Kr \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta g(Kr \sin \theta) d\theta$$

distribution of $\omega_k$

$g = g(\omega_k)$

References:
J. Acebron et al.. Rev. Mod. Phys. 77, 137 (2005)
The Kuramoto solution (cont.)

e.g. For Lorentz distribution \( g(\omega) = \frac{\Delta}{\pi[\Delta^2 + (\omega)^2]} \)

the solution is \( r = \sqrt{1 - 2\Delta/K} \)

Solution describes self-synchronizing “phase transition”

Solution valid

- when system is in steady state
- in thermodynamic limit (\( N \to \infty \))
- for global coupling

Scaling:

\[ r \propto \sqrt{K - K_c} \]

Partial sync (not fully locked!)

phase transition at \( K_c = 2\Delta \)
Motivation for linear reformulation

• Linear reformulation of the nonlinear synchronization problem

  – Apply linear methods such as spectral analysis to generalize and gain intuition about spontaneous sync

    Finite N
    Different coupling schemes

  – New way of looking at this problem: The spectrum completely determines sync properties

• Explore nonequilibrium phase transitions
Outline

• Linear reformulation
• Mapping between linear and Kuramoto models
• Examples of finite case
• Extension of linear model to continuum systems
• New classes of exactly solvable models in the continuum limit and their properties
• Summary
The linear reformulation – finite system

Model:

\[
\dot{\psi}_k = (i\omega_k - \gamma)\psi_k + \sum_{j \neq k}^{N} \Omega_{jk} \psi_j
\]

- decay constant assumed > 0
- coupling constant assumed real

\[
\psi_k(t) = R_k(t)e^{i\theta_k(t)}
\]

- maps onto Kuramoto model and synchronizes
- constrained by \( \gamma \) to go to a steady state value

Note: Only dealing with phase synchronization in this work
The linear reformulation – finite system (cont.)

Solution:

\[ \vec{\psi} = \sum_{j=1}^{N} a_j \vec{v}_j e^{\lambda_j t} \]

- eigenvectors
- determined by initial conditions
- eigenvalues
- \( \lambda_1, \ldots, \lambda_N \)
  - smallest real part
  - largest real part
- Tune \( \gamma \) for \( \Re[\lambda_N] = 0 \)
Synchronization in the linear model

If \( \Re[\lambda_{N-1}] \neq \Re[\lambda_N] \),

then \( \lim_{t \to \infty} \vec{\psi} = a_N \vec{\psi}_N e^{i\omega_r t} \) => fully phase locked

collective freq. of synchronized state: \( \omega_r \equiv \phi/t = -i\lambda_N = |\lambda_N| \)

\[
r \equiv \left| \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \right| \quad = \frac{1}{N} \left| \sum_{j=1}^{N} \frac{\psi_j}{|\psi_j|} \right| \quad = \frac{1}{N} \left| \sum_{j=1}^{N} \frac{(\nu_N)_j}{|(\nu_N)_j|} \right|
\]

Independent of initial conditions
Synchronization in the linear model (cont.)

Assuming \( g(\omega_r - \omega_k) = g(\omega_r + \omega_k) \)

and \( \Omega_{jk} = \frac{\Omega}{N} > 0 \)

then from \( \vec{u}_N \) (eigenvector associated with \( \lambda_N \)),

\[
    r = \frac{1}{N} \sum_{j=1}^{N} \left[ 1 + \left( \frac{\omega_j - \omega_r}{\Omega/N + \gamma} \right)^2 \right]^{-1/2}
\]

Explicit solution of order parameter given \( \omega_k, \Omega \)

Valid for any \( N \)
The “phase transition” for finite $N$

$r$ has a nonzero steady-state value

$r$ does not have a nonzero steady-state value

Defined by $\Re[\lambda_N] = \Re[\lambda_{N-1}]$, e.g.

Partial sync:

Discusses only fully locked transition

$\frac{1}{\sqrt{N}}$
The “phase transition” for finite $N$

- $r$ has a nonzero steady-state value
- $r$ does not have a nonzero steady-state value

Defined by $\mathcal{R}[\lambda_N] = \mathcal{R}[\lambda_{N-1}]$, e.g.

Since $\mathcal{R}[\lambda_N] = 0$ by construction (by tuning $\gamma$), the critical point occurs when $\mathcal{R}[\lambda_{N-1}(\Omega_c)] = 0$

Critical value of coupling constant

A true thermodynamic phase transition occurs as $N \to \infty$ (Discussed later)
Mapping the linear model to the Kuramoto model

Linear model: $\ddot{\psi}_k = (i\omega_k - \gamma)\psi_k + \sum_{j \neq k}^{N} \Omega_{jk} \psi_j$

Nonlinear transformation: $\psi_k(t) = R_k(t)e^{i\theta_k(t)}$

$\dot{R}_k(t) = -\gamma R_k + \sum_{j \neq k}^{N} \Omega_{jk} R_j \cos(\theta_j - \theta_k)$

$\dot{\theta}_k = \omega_k + \sum_{j \neq k}^{N} \Omega_{jk} \frac{R_j}{R_k} \sin(\theta_j - \theta_k)$

= Kuramoto model, if $R_k$ goes to a steady state as $t \to \infty$
Mapping the linear model to the Kuramoto model (cont.)

Assuming $\Omega_{jk} = \Omega/N$, the linear model maps onto the Kuramoto model with the following effective coupling constant:

$$
\tilde{K}_{jk} = \lim_{t \to \infty} \frac{\Omega}{N} \frac{R_j}{R_k} = \frac{\Omega}{N} \sqrt{\frac{(\omega_k - \omega_r)^2 + (\Omega/N + \gamma)^2}{(\omega_j - \omega_r)^2 + (\Omega/N + \gamma)^2}}
$$

independent of initial conditions

Open problem:
More flexible mappings such as $\gamma \rightarrow \gamma_k$, $\Omega \rightarrow \Omega_{jk}$, etc. preserve linearity of problem

What general class of coupling $\tilde{K}_{jk}$ can this linear reformulation be used for?
Example 1

**Case**: $\omega_k = \omega$  
System $\rightarrow$ full synchronization unless $\Omega = 0$

<table>
<thead>
<tr>
<th>N-1 degenerate eigenvalues</th>
<th>1 unique eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{deg} = i\omega - \gamma - \Omega/N$</td>
<td>$\lambda_N = i\omega - \gamma + \Omega(N - 1)/N$</td>
</tr>
<tr>
<td>$v_N = {1, 1, 1, \ldots, 1}$</td>
<td>$v_N = {1, 1, 1, \ldots, 1}$</td>
</tr>
</tbody>
</table>

| $\gamma = \Omega(N - 1)/N$ | degenerate eigenvectors decay | $|\psi_k| = R_k \rightarrow$ steady state |

Synchronization timescale $\frac{N}{(N-2)\Omega} \sim \frac{1}{\Omega}$ for large $N$
Example 1 (cont.)

Assuming $\Omega \neq 0$

**Linear solution**

$$\lim_{t \to \infty} \vec{\psi} = a_N \vec{v}_N e^{i\omega t}$$

finite $N$

$r = 1$

**Kuramoto solution**

As $t \to \infty$, $\tilde{K}_{jk} \to \Omega/N$

$$g(\omega) = \delta(\omega - \omega_k)$$

**Small perturbations:** $\omega_k = \omega + \epsilon \eta_k$

$$r \approx 1 - \epsilon^2 \frac{\Delta^2}{2\Omega^2}$$

where 

$$\Delta^2 = \frac{1}{N} \sum_{j=1}^{N} \eta_j^2$$

variance
Example 2

Case:

<table>
<thead>
<tr>
<th>Bimodal distribution</th>
<th>(Assume N is even)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \leq j \leq N/2$</td>
<td>$N/2 &lt; j \leq N$</td>
</tr>
<tr>
<td>$\omega_j = \omega_1$</td>
<td>$\omega_2 \geq \omega_1$</td>
</tr>
<tr>
<td>$\omega_1 \leq \omega_2$</td>
<td></td>
</tr>
</tbody>
</table>

(N-2)/2 degenerate eigenvalues

$\lambda_{deg_1} = i\omega_1 - \Omega/N - \gamma$

$\lambda_{deg_2} = i\omega_2 - \Omega/N - \gamma$

2 unique eigenvalues

$\lambda_{N,N-1} = i\frac{\omega_1 + \omega_2}{2} - \gamma + \frac{\Omega(N-2)}{2N} \pm \frac{\sqrt{\Omega^2 - \Delta^2}}{2}$

where $\Delta = |\omega_2 - \omega_1|$
Example 2 (cont.)

Situation 1: $\Omega < \Delta$

$$\lambda_{N,N-1} = i \left( \frac{\omega_1 + \omega_2 \pm \sqrt{\Delta^2 - \Omega^2}}{2} \right) - \gamma + \frac{\Omega(N-2)}{2N}$$

$$\gamma = \frac{\Omega(N-2)}{2N}$$

$$\lim_{t \to \infty} \tilde{\psi} = a_{(N-1)} \tilde{v}_{(N-1)} e^{i\left( \frac{\omega_1 + \omega_2 - \sqrt{\Delta^2 - \Omega^2}}{2} \right)t} + a_N \tilde{v}_N e^{i\left( \frac{\omega_1 + \omega_2 + \sqrt{\Delta^2 - \Omega^2}}{2} \right)t}$$

$\rightarrow$ $r$ will never go to a steady state

$\rightarrow$ No steady-state synchronization
Example 2 (cont.)

Situation 2: $\Omega > \Delta$

$$\gamma = \frac{\Omega(N-2)}{2N} + \frac{\sqrt{\Omega^2 - \Delta^2}}{2} \quad \rightarrow \quad \text{N-1 eigenvectors decay on a time scale } \frac{N}{(n-2)\Omega}$$

$$\lim_{t \to \infty} \psi = a_N \vec{v}_N e^{i\omega_1 + \omega_2 t}$$

Remaining eigenvector:

$$(v_N)_j = \begin{cases} 
\frac{i\Delta - 2\Omega/N - \sqrt{\Omega^2 - \Delta^2}}{-i\Delta - 2\Omega/N - \sqrt{\Omega^2 - \Delta^2}} & 1 \leq j \leq N/2 \\
1 & N/2 < j \leq N 
\end{cases}$$

$$r = \sqrt{\frac{1 + \sqrt{1 - \left(\frac{\Delta}{\Omega}\right)^2}}{2}}$$

Independent of $N$ \quad Valid for any $N$
Example 2 (cont.)

<table>
<thead>
<tr>
<th>No synchronization</th>
<th>Steady state synchronization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega &lt; \Delta$</td>
<td>$\Omega &gt; \Delta$</td>
</tr>
<tr>
<td>$\Omega_c = \Delta$</td>
<td></td>
</tr>
<tr>
<td>1st order “phase transition”</td>
<td></td>
</tr>
<tr>
<td>$r_c = \frac{1}{\sqrt{2}}$</td>
<td></td>
</tr>
</tbody>
</table>

Square-root scaling law near phase transition:

$$r - r_c \approx \frac{1}{2} \sqrt{\frac{\Omega - \Omega_c}{\Omega}}$$

$t \to \infty$  \(\tilde{K}_{jk} \to \Omega/N\)

Linear solution  \(\Omega \to K\)  Kuramoto solution

Assuming \(g(w) = \frac{\delta(w-\Delta/2)}{2} + \frac{\delta(w+\Delta/2)}{2}\)

finite N  \(\lim_{N \to \infty}\)

\[r = \sqrt{1 + \sqrt{1 - \left(\frac{\Delta}{\Omega}\right)^2}}\]
The linear reformulation – continuum limit

Recall:

\[ \psi_k = (i \omega_k - \gamma) \psi_k + \sum_{j \neq k}^N \Omega_{jk} \psi_j \]

\[ N \to \infty: \quad \dot{\psi}(\omega, t) = (i \omega - \gamma) \psi(\omega, t) + \int_{-\infty}^{\infty} \Omega(\omega, \omega') g(\omega) \psi(\omega, t) d\omega \]
The linear reformulation – continuum limit

Recall: \[ \dot{\psi}_k = (i\omega_k - \gamma)\psi_k + \sum_{j \neq k}^{N} \Omega_{jk} \psi_j \]

\[ N \rightarrow \infty: \dot{\psi}(\omega, t) = (i\omega - \gamma)\psi(\omega, t) + \int_{-\infty}^{\infty} \Omega(\omega, \omega') g(\omega) \psi(\omega, t) d\omega \]

Mapping: \[ \psi(\omega, t) = R(\omega, t) e^{i\theta(\omega, t)} \]

Constrained by \( \gamma \) to go to a steady state value

Maps onto Kuramoto synchronization model

\[ \dot{R}(\omega, t) = -\gamma R(\omega, t) + \int_{-\infty}^{\infty} \Omega(\omega, \omega') g(\omega') R(\omega', t) \cos[\theta(\omega', t) - \theta(\omega, t)] d\omega' \]

\[ \dot{\theta}(\omega, t) = \omega + \int_{-\infty}^{\infty} \Omega(\omega, \omega') \frac{R(\omega', t)}{R(\omega, t)} g(\omega') \sin[\theta(\omega', t) - \theta(\omega, t)] d\omega' \]

Effective coupling \( \tilde{K}(\omega, \omega', t) \) for Kuramoto model

Becomes Kuramoto model with t-independent coupling if \( R(\omega, t) \) goes to a steady state
Linear solution in the continuum limit for a general symmetric distribution $g(\omega)$.

Assuming $g(\omega_r - \omega_k) = g(\omega_r + \omega_k)$

and $\tilde{K}(\omega, \omega') = \Omega(\omega, \omega') \sqrt{\frac{(\omega - \omega_r)^2 + \gamma^2}{(\omega' - \omega_r)^2 + \gamma^2}}$

• $r = \int_{-\infty}^{\infty} d\omega g(\omega) \left[ 1 + \left( \frac{\omega - \omega_r}{\gamma} \right)^2 \right]^{-1/2}$

• Self-consistency equation: $1 = \Omega \int_{-\infty}^{\infty} d\omega g(\omega) \frac{\gamma}{\gamma^2 + \omega^2}$

where $\Omega(\omega, \omega') = \Omega$

Exactly solves system with a given $g(\omega)$ with $\tilde{K}(\omega, \omega')$
Scaling of linear solution about \( \Omega_c \) for \( g(\omega) \).

Recall \( \Omega_c = \frac{1}{\pi g(0)} \)

Anomalous critical scaling:

As \( \Omega \to \Omega_c \),

\[
\gamma = -\frac{\pi g(0)}{\int_{-\infty}^{\infty} d\omega \frac{g'(\omega)}{\omega}} \left( \frac{\Omega - \Omega_c}{\Omega_c} \right) + O \left( \left( \frac{\Omega - \Omega_c}{\Omega_c} \right)^2 \right)
\]

and as \( \gamma \to 0 \),

\[
r(\gamma) = -2g(0)\gamma \log[g(0)\gamma] + O[g(0)\gamma]
\]

May be due to bicritical point, i.e. 2 phase transitions occurring at the same point

incoherent - partially locked  \hspace{1cm}  partially locked - fully locked
Spectrum of the operator, $\hat{L}$, of the linear model

$$\dot{\psi}(\omega, t) = \hat{L}(\omega)\psi(\omega, t)$$
What is happening?

$\Omega < \Omega_c$

$r = 0$

$\gamma = 0$

$\text{Im} \lambda$

$\text{Re} \lambda$
What is happening?

\[ \Omega < \Omega_c \]

\[ r = 0 \]

“Splay state” \[ r = 0 \]

Fully locked and sync transition
Anomalous scaling

\[ \Omega = \Omega_c \]
What is happening?

For $\Omega < \Omega_c$:

$r = 0$

For $\Omega = \Omega_c$:

“Splay state” $r = 0$

$\gamma = 0$

For $\Omega > \Omega_c$:

$r \approx 1$

$\gamma \rightarrow$

Fully locked and sync transition
Anomalous scaling
Examples: 2 distributions for $K(\omega, \omega') = \Omega \sqrt{\frac{(\omega-\omega_r)^2 + \gamma^2}{(\omega'-\omega_r)^2 + \gamma^2}}$

**Lorentzian distribution** about $\omega_r$

$$g(\omega - \omega_r) = \frac{\Delta}{\pi[\Delta^2 + (\omega - \omega_r)^2]}$$

$$\Omega_c = \Delta$$

Synchronization time:

$$\gamma_{lor}^{-1} = \frac{1}{\Omega - \Delta}$$

Linear solution:

For $\Omega > \Omega_c$,

$$r_{lor} = \frac{2 \cos^{-1}\left(\frac{\Omega_c}{\Omega - \Omega_c}\right)}{\pi \sqrt{1 - \left(\frac{\Omega_c}{\Omega - \Omega_c}\right)^2}}$$

**Uniform distribution** about $\omega_r$

$$g(\omega - \omega_r) = \begin{cases} \frac{1}{\pi \Delta} & |\omega - \omega_r| \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Omega_c = \Delta$$

Synchronization time:

$$\gamma_{unif}^{-1} = \frac{2}{\Delta \pi} \tan\left(\frac{\pi \Delta}{2 \Omega}\right)$$

Linear solution:

For $\Omega > \Omega_c$,

$$r_{unif} = \cot\left(\frac{\pi \Omega_c}{2 \Omega}\right) \sinh^{-1}\left[\tan\left(\frac{\pi \Omega_c}{2 \Omega}\right)\right]$$
Continuum examples (cont.)

\[ \Omega / \Omega_c, \frac{K}{K_c} \]

- Linear \( r_{unif} \)
- Kuramoto \( r_{lor} \)
- Linear \( r_{lor} \)
# Kuramoto model vs. linear reformulation

<table>
<thead>
<tr>
<th>Kuramoto model</th>
<th>Linear reformulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>• is nonlinear</td>
<td>• is linear</td>
</tr>
<tr>
<td>• has an exact (implicit) sync solution that is only</td>
<td>• has an exact (explicit) sync solution that is</td>
</tr>
<tr>
<td>– valid at $N \to \infty$</td>
<td>– valid at any $N$</td>
</tr>
<tr>
<td>– for uniform global coupling</td>
<td>– adaptable to other coupling schemes</td>
</tr>
<tr>
<td>– in steady state</td>
<td>– dynamics are solvable</td>
</tr>
<tr>
<td></td>
<td>• opens up new class of exactly solvable models in the continuum limit</td>
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</tbody>
</table>

D.C.R. PRE 77, 031114 (2008)
http://cnls.lanl.gov/~dcr/
Summary

- Linear dynamics reformulation of Kuramoto model
- Spectrum determines synchronization properties (order parameter, critical point, etc.)

Outlook

1. What is the general class of $\tilde{K}(\omega, \omega')$ for which this linear reformulation works?

2. Can the spectral method be applied to partial synchronization problems?