Problems for ‘DIAMANT meets GQT’

(in arbitrary order)

Contents

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1 Diffeomorphisms of the interval and conformal field theory

Problem proposed by André Henriques (UU)

Statement of Problem

Let $\mathrm{Diff}([0,1])$ denote the group of diffeomorphisms of the interval, and let $\mathrm{Diff}'([0,1]) \subset \mathrm{Diff}([0,1])$ be the subgroup of diffeomorphisms that fix the tangent vectors at the two end points. Question: compute the group cohomology of these two groups.

For a less ambitious question: compute the universal central extensions of these two groups, and relate them to the universal central extension of $\mathrm{Diff}(S^1)$.

Background information

In its current formulation (due to Graeme Segal), conformal field theory is a $1+1$ dimensional field theory. In other words, it assigns algebraic objects of closed 1-dimensional manifolds, and to 2-dimensional manifolds with boundary. Refining the definition of CFT to a $0{+}1{+}1$ theory is an intersting and difficult open problem. Roughly speaking, it would require having algebraic objects associated to intervals, and to 2-dimensional manifolds with corners. A good understanding of the group $\mathrm{Diff}([0,1])$ seems crucial for any progress in that direction.

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2 Fundamental domains for nonuniform lattices in Kac–Moody groups

Problem proposed by Lisa Carbone (Rutgers)

Statement of Problem

Let $G = G_A(\mathbb{F}_q)$ be a locally compact complete Kac-Moody group over a finite field $\mathbb{F}_q$, corresponding to a generalized Cartan matrix A. Let $\Gamma \leq G$ be a non-uniform lattice subgroup (that is, Γ is a discrete subgroup for which $\Gamma\backslash G$ is not compact, yet carries a finite G-invariant measure), known to exist by work of Carbone and Garland and by Rémy. Let X denote the Tits building of G. If $rank(G) = 2$ then X is the homogeneous tree X_{q+1}. The following conjecture was posed by Howard Garland.

Conjecture: The quotient $\Gamma\backslash X$ of X by Γ consists of a finite core graph together with finitely many cusps which are semi-infinite rays.

Background information

This conjecture holds for all known lattice subgroups of G ([CCM]). In particular if G is of affine type and G is completed using the Rémy-Ronan method then we can identify G with $SL_2(\mathbb{F}_q((t)))$. The conjecture holds here by the work of Lubotzky ([L]) and also by Ragunathan ([R]).

An analog of the conjecture also holds for $\mathbb{R}$-rank one, $\mathbb{R}$-simple Lie groups, where Garland and Raghunathan showed that fundamental domains for nonuniform lattices are finite unions of Siegel sets ([GR]).

A positive answer to the conjecture for rank 2 hyperbolic Kac-Moody groups would have a number of important consequences. For example, it would imply the 'Kazhdan-Margulis' property for Kac-Moody groups, namely that covolumes of lattices are bounded away from zero.

It would also have consequences for the study of automorphic forms on arithmetic quotients of Kac-Moody groups, a subject that is not yet well understood. It is known that the automorphic spectrum of the combinatorial Laplacian with respect to a nonuniform lattice Γ acting on a homogeneous tree X is determined by the 'edge-indexed' quotient graph of Γ on X. If G is the projective affine Kac-Moody group $PGL_2(\mathbb{F}_q((t)))$, Efrat ([E]) used this to study Eisenstein series on quotients of X by the lattice $\Gamma = PGL_2(\mathbb{F}_q[t])$ and its congruence subgroups.

To develop an analogous theory for hyperbolic Kac-Moody groups, it is desirable then to obtain a structure theorem for quotients of the Tits building by nonuniform lattices.

References

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3 Witten index of the independence complex of 2d-grids

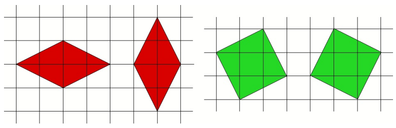
Problem proposed by Liza Huijse and Kareljan Schoutens (UvA)

Statement of Problem

The challenge is to compute the cohomology of the independence complex for specific 2-dimensional grids, the triangular and hexagonal grids with periodic boundary conditions in particular. A derived result will be the value of the Euler characteristic. (An independent set in a simple and loopless graph G is a subset of the vertex set of G with the property that no two vertices in the subset are adjacent. The family of independent sets in G forms a simplicial complex, the independence complex of G.)

Background information

Recent work by J. Jonsson (Stockholm) [JON] and by L. Huijse, J. Halverson, P. Fendley and K. Schoutens [HHFS] has led to the full solution for the 2-dimensional square grid. The cohomology cycles turn out to be directly related to tilings of the plane with two types of rhombi (see figure). The dimension of the cohomology grows exponentially with the linear dimensions of the grid.



For certain classes of 2-grids, including the triangular and hexagonal grids, an upper bound to the dimension of the cohomology was obtained [ENG]. It seems clear that for the triangular and hexagonal grids the dimension of the cohomology grows exponentially with the area of the grid. For these grids a set of rational numbers r has been determined, such that there exist cross-cycles of size rN, where N is the number of vertices of the 2-grid. These cross-cycles are specific cohomology cycles that are given in closed form [JON]. For the triangular grid $r \in [\frac{1}{7}, \frac{1}{5}] \cap \mathbb{Q}$ and for the hexagonal grid $r \in [\frac{1}{4}, \frac{5}{18}] \cap \mathbb{Q}$. The latter results suggest that a relation between the cohomology cycles and tilings of the plane may also exist for the triangular and hexagonal grids. A first step towards obtaining a lower bound to the dimension of the cohomology for the triangular grid using this relation has been made. Progress in this direction seems within reach.

In the physics context [FS] [HS], the cohomology cycles correspond to ground states of supersymmetric lattice models of strongly correlated fermions. The Euler characteristic is the Witten index. These models exhibit so-called superfrustration: there are an exponential number of ways in which the system can realise its minimal energy. This property follows directly from the growth behaviour of the dimension of the cohomology. The rational numbers r given above, have the interpretation of filling fraction (number of particles per vertex). Finally, the relation with tilings allows for a very nice interpretation of excitations in these models.

References

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4 Flatness and deformation theory

Problem proposed Bart de Smit (UL)

Statement of Problem

Let A in B be a finite flat local extension of local artin rings with the same residue field $k = B/m_B = A/m_A$. Suppose that A and B have the same embedding dimension, i.e. $\dim_k(m_A/m_A^2) = \dim_k(m_B/m_B^2)$. Then is it true that every finitely generated B-module that is flat as an A-module is flat as a B-module?

Background information

Problem came up in connection with the proof of Fermat's Last Theorem by Wiles. It would be a very strong property if true. For embedding dimension one, it is true: $A = k[X]/(X^n)$ and $B = A[Y]/(Y^m - X)$. For embedding dimension two, there is a proof by Brochard and Mézard.

References

Sylvain Brochard and Ariane Mézard, *About De Smit's conjecture on flatness* (Draft version),
`http://www.math.leidenuniv.nl/~brochard/desmitconj.pdf`

5 Growth of groups

Problem proposed by Aner Shalev (HUJI)

Statement of Problem

Let p be a prime. For a polynomial $Q \in \mathbf{F}_p[t]$ that is relatively prime to both t and $1+t$ let o_Q be the order of the multiplicative group generated by t and $1+t$ modulo Q. Question: how does the cardinality

$$f(x) := |\{o_Q \ : \ Q \text{ relatively prime to } t, 1+t\} \cap [1,x]|$$

grow with x? Aner Shalev hopes that for all p there exists an $\varepsilon > 0$ such that $f(x) > x^{\varepsilon}$ for x sufficiently large.

Background information

The problem arose in the context of some (yet unpublished) work with Martin Liebeck on dimensions of irreducible representations of infinite groups.

6 Combinatorial explanation of polynomiality of Hurwitz numbers

Problem proposed by Ravi Vakil (Stanford)

Statement of Problem

Given a permutation in conjugacy class α, in how many ways can it be factored into r transpositions that "connect" the numbers 1 through n (*transitive* factorizations). (If the condition of transitivity seems unnatural, it is straightforward to connect to the problem without the transitive condition.)

Background information

This problem is not related to the topic of my talk, but if solved, would be very important. It is a purely discrete/combinatorial problem, but I'll begin with the geometric version. Fix a genus g, a degree d, and a partition of d into n parts, $\alpha_1 + \cdots + \alpha_n = d$, and let $r = 2g+d+n-2$. Fix $r+1$ distinct points $p_1, \ldots, p_r, \infty$ on $\mathbb{CP}^1$. Define the *Hurwitz number* H^g_α as the number of branched covers of $\mathbb{CP}^1$ by a (connected) Riemann surface, that are unbranched away from $p_1, \ldots, p_r, \infty$, such that the branching over ∞ is given by $\alpha_1, \ldots, \alpha_n$ (the monodromy lies in the conjugacy class corresponding to that partition), and the branching over each p_i is $2+1+\cdots+1 = d$ (the simplest nontrivial branching). We consider the n preimages of ∞ to be labeled.

Up to a straightforward combinatorial factor, the Hurwitz number corresponds to the answer to the combinatorial problem. (If the condition of transitivity seems unnatural, it is straightforward to connect to the problem without the transitive condition. This is equivalent to counting potentially disconnected covers. The algebraically simplest way to relate them: the exponential of the generating function counting connected covers is the generating function counting potentially disconnected covers.)

Based on extensive evidence, the combinatorialists Goulden and Jackson had conjectured that this combinatorial problem had a surprising polynomial behavior: fixing g and n, H^g_α is a simple combinatorial term times a symmetric polynomial in $\alpha_1, \ldots, \alpha_n$, with components in homogeneous degree between $2g-3+n$ and $3g-3+n$. This strongly suggests a connection between this combinatorial problem and the moduli space of curves (to those familiar with the latter)!

The only known proof of this fact goes through the moduli space of curves. Is a direct combinatorial argument possible?

Ekedahl, Lando, M. Shapiro, and Vainshtein explained this polynomiality with their ground-breaking ELSV-formula. Through the ELSV formula, the highest-degree terms are seen to be precisely the subject of Witten's conjecture, and most of the proofs of Witten's conjecture go through this combinatorial gadget. The lowest-degree terms are the subject of the "λ_g-conjecture", also now a theorem.

References

For more details, including lots of references, see the two surveys:

Y.-P. Lee and R. Vakil, *Algebraic structures on the topology of the moduli spaces of curves and maps*, `http://math.stanford.edu/~vakil/preprints.html`, earlier version arXiv:0809.1879.
R. Vakil, *The moduli space of curves and Gromov-Witten theory*, in *Enumerative Invariants in Algebraic Geometry and String Theory* (Behrend and Manetti eds.), Lecture Notes in Math., Springer-Verlag, 2008.

7 Sums of negative powers modulo a prime power

Problem proposed by Rob de Jeu (VU)

Statement of Problem

For $n \geq 3$ an odd integer, p an odd prime number, and $m \geq 1$, we define in $\mathbb{Z}/p^m\mathbb{Z}$,

$$S(n,p,m) = \sum_{\substack{a=1 \\ \gcd(a,p)=1}}^{p^m} (-1)^a (\overline{a})^{-n}\,.$$

(a) For given n, are there infinitely many p such that $S(n,p,1) \neq \overline{0}$?
(b) For given n and p, does there exist $m \geq 1$ with $S(n,p,m) \neq \overline{0}$?
(c) For fixed n and p, the $S(n,p,m)$ are the reductions of an element in $\mathbb{Z}_p$. Is this element irrational?

Background information

(a) and (b) have been checked in many cases. For example, $S(n,3,1) = -(\overline{1})^{-n} + (\overline{2})^{-n} = \overline{1}$ because n is odd, so $S(n,3,m) \neq \overline{0}$ for all $m \geq 1$. For $2 < p < 10^5$ and $3 \leq n \leq 19$, $S(n,p,1) \neq \overline{0}$ except when $(n,p) = (3,16843)$, $(5,37)$, $(9,17)$, $(9,67)$, $(9,877)$, $(11,31)$, $(11,9311)$, $(11,42019)$, $(15,43)$, $(15,59)$, $(15,127)$, $(15,607)$, $(17,257)$, $(17,2591)$, $(19,73)$, $(19,149)$, $(19,311)$, $(19,401)$ or $(19,10133)$, and $S(n,p,2) \neq \overline{0}$ in all those cases. Similarly, $S(n,37,5) = \overline{0}$ but $S(n,37,6) \neq \overline{0}$ for $n = 11692013 = 5 + 36(29 + 8 \cdot 37 + 15 \cdot 37^2 + 6 \cdot 37^3)$. The very explicit questions are motivated by a relation between values of p-adic L-functions and p-adic polylogarithms (see [5, (4) on page 173] or [1, page 405]),

$$(1) \qquad (1 - 2^{n-1}) L_p(n, \omega_p^{1-n}, \mathbb{Q}) = 2^{n-1} \mathrm{Li}_n^{(p)}(-1) \qquad (n \geq 3 \text{ and odd})\,,$$

where ω_p is the Teichmüller character for $\mathbb{Q}$ and p, and $\mathrm{Li}_n^{(p)}(-1)$ describes the syntomic regulator on $K_{2n-1}(\mathbb{Q})$ [2, Theorem 1.12]. By [6, V §1], $2\mathrm{Li}_n^{(p)}(-1)$ is in $\mathbb{Z}_p$ and reduces to $S(n,p,m)$ in $\mathbb{Z}/p^m\mathbb{Z}$ for $m \geq 1$, $p \geq 3$ and $n \geq 1$. Thus (a) states that for fixed odd $n \geq 3$, both sides in (1) are in $\mathbb{Z}_p^*$ for infinitely many primes p, and (b) that they are always non-zero. Since certain values of p-adic L-functions are irrational [4, 3], (c) is a natural question. If $p-1$ divides $n-1$, then the p-adic L-function has no zeroes, so (b) holds in this case. If $p-1$ does not divide $n-1$ and $p > n \geq 3$, then, with B_i the i-th Bernoulli number, $L_p(n, \omega_p^{1-n}, \mathbb{Q}) = \lim_{a\to\infty} L_p(n-(p-1)p^a, \omega_p^{1-n}, \mathbb{Q}) \equiv -\frac{B_{p-n}}{p-n}$ modulo p by Theorems 5.11 and 4.2 as well as Corollary 5.14 of [7]. This explains why when $S(n,p,1) = \overline{0}$, p is often an irregular prime (in fact, such that the numerator of B_{p-n} contains a factor p).

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8 Gasca's conjecture

Problem proposed by Jan Draisma (TU/e)

Statement of Problem

Suppose $n := \binom{d+2}{2}$ points $p_1, \ldots, p_n$ in $\mathbb{R}^2$ have the following property: for all $i = 1, \ldots, n$ there exist d straight lines $l_{i,1}, \ldots, l_{i,d}$ whose union contains all p_j with $j \neq i$, but not p_i. A conjecture by Gasca states that then there exists a line containing $d+1$ points. Prove this conjecture.

Background information

- Let us say that the p_i are in *Gasca position* if they satisfy the hypothesis. This is not standard terminology.
- One configuration of points in Gasca position is the set of intersections of $d+2$ lines in general position. Of course, this configuration also satisfies the conclusion. However, this is not the only type of configuration in Gasca position; see Figure 1 for some examples with $d = 3$.
- The number n is the dimension of the space of polynomials of degree $\leq d$ in 2 variables. So the hypothesis implies that the polynomials $f_i := \prod_{k=1}^{d} l_{i,k}$, where we identify the lines with their defining affine-linear forms, form a basis of this space. This has a number of easy consequences. For instance, for all i the $l_{i,k}$ are distinct and unique up to permutation.
- The question makes sense with $\mathbb{R}^2$ replaced by any other affine or projective plane. As far as I know no counterexamples have been found. The statement does not hold in spaces where the only axiom is that through two points goes at most one line (Blokhuis and Brouwer found a counterexample with $d = 4$).
- If the conjecture is true, then removing $d+1$ points on a line leads to a configuration with the same property for $d-1$ instead of d.
- The conjecture has been proved, for $\mathbb{R}^2$, for d up to 4.
- The conjecture comes from multivariate interpolation: a configuration of points in Gasca position allows for easy Lagrange-type interpolation formulas.

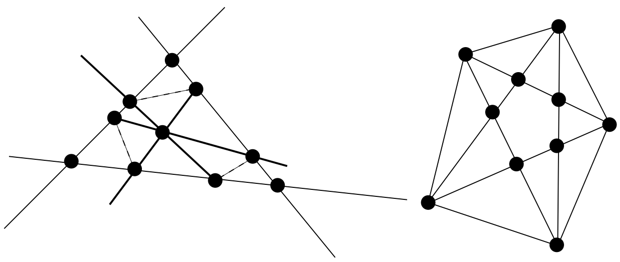

Figure 1: Two configurations in Gasca position with $d = 3$.

References

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9 Diffeomorphisms groups and renormalization

Problem proposed by Walter van Suijlekom (RU)

Statement of Problem

Consider the group G of formal diffeomorphism in n variables $x_1, ..., x_n$. Let I be an ideal in $k[[x_1, ..., x_n]]$ and let G^I be the subgroup of G that leaves I invariant.

Question: What are the conditions on I so that G^I is isomorphic to a group of formal diffeomorphisms (in say $m \leq n$ variables)?

A simplification of the problem would be to first consider polynomial ideals, i.e. $I \subset k[x_1, \ldots, x_n]$ and study the structure of the group G^I.

Background information

The motivation for this problem comes from gauge theories in physics. In (perturbative) quantum Yang–Mills gauge theories, one assigns coupling constants $x_1, \ldots, x_n$ to measure the strength of different physical interactions. Although the name might suggest otherwise, the coupling constants depend on the energy. This dependence is nicely described by means of an action on the coupling constants by the renormalization group (see for instance the lecture by D. Gross on renormalization groups in [2]. The renormalization group turns out to be (related to) the group G of formal diffeomorphism, acting on the variables $x_1, \ldots, x_n$. This was derived in [1] for the simplest case of one coupling constant and in [3] for the general case.

Now, gauge theories are special in that they possess a *gauge symmetry* (cf. the lecture by L. Faddeev on Yang–Mills fields in [2]). These symmetries imply that there are relations between the coupling constants $x_1, \ldots, x_n$. In fact, in the case of a Yang–Mills gauge theory with simple gauge Lie group, it turns out [3] that the relations are just $x_1 = x_2 = \cdots = x_n$. This implies that the subgroup of G that respects these relations is the formal diffeomorphism group in one variable. In physics, this fact is of great importance in establishing renormalizability of gauge theories.

In the case of a general gauge theory, there will be some polynomial relations between the coupling constants, but it is not clear what the subgroup of G is that respects these relations.

References

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10 Odd polynomials and the construction of non-tame automorphisms

Problem proposed Stefan Maubach (RU)

Statement of Problem

Let K denote a field. We say that a vector of n-variable polynomials

$$F = (F_1(X_1, \ldots, X_n), \ldots, F_n(X_1, \ldots, X_n)) \in K[X_1, \ldots, X_n]^n$$

is a *polynomial automorphism* if there exists $G \in K[X_1, \ldots, X_n]^n$ with $F \circ G = I$. Here, "$\circ$" denotes composition of polynomials, and I denotes the "identity polynomial automorphism $I = (X_1, \ldots, X_n)$. Note that polynomial automorphisms include all invertible linear maps (or "nonsingular matrices" if you like that more). Just to give you one example of a "nontrivial" polynomial automorphism in two variables: $(X + Y^2, Y)$ has inverse $(X - Y^2, Y)$.

Question. Do there exist odd polynomial automorphisms over the finite fields $\mathbf{F}_{2^k}$ for $k \geq 2$? Here, an n-variable polynomial automorphism is seen as a permutation of q^n elements, and called odd if the corresponding permutation is so.

Background information

It is known that over a finite field of odd characteristic, and over $\mathbf{F}_2$, every permutation arises from a polynomial automorphism, and over $\mathbf{F}_{2^k}$, every even permutation arises from a polynomial automorphism.

Applications are the construction of non-tame automorphisms, and disproving certain conjectures about generators for groups of affine maps. See the second reference below.

References

Stefan Maubach, *Polynomial automorphisms over finite fields*, Serdica Math. J. **27** (2001) no.4. 343-350.

Stefan Maubach, *A problem on polynomial maps over finite fields*, `arxiv.org:0802.0630`

11 100 Digits Challenge

Problem proposed by Frank Vallentin (CWI Amsterdam)

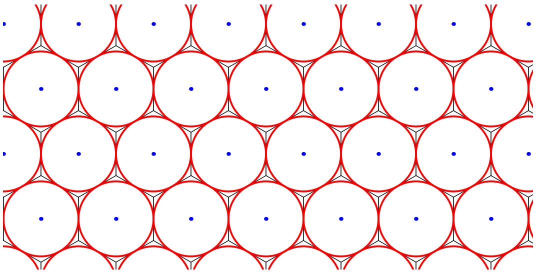
Statement of Problem

Let $L_k^\alpha(x)$ be the Laguerre polynomials, i.e. polynomials which are orthogonal for the inner product $(f, g) = \int_0^\infty f(x)g(x)e^{-x}x^\alpha dx$, and normalized by $L_k^\alpha(0) = 1$. Set $\alpha = n/2 - 1$. For $n = 2, 8, 24$, compute the the first ***100 digits*** of the optimal value of the infinite dimensional linear program in the optimization variables $\alpha_0, \alpha_1, \ldots$

$$\sup \Big\{ \sum_{k=0}^{\infty} (-1)^k \alpha_k : \sum_{k=0}^{\infty} \alpha_k = 1, \quad \sum_{k=0}^{\infty} \alpha_k L_k^\alpha(2\pi x^2) \leq 0, \quad x \geq 1, \quad \sum_{k=0}^{\infty} (-1)^k \alpha_k L_k^\alpha(2\pi x^2) \geq 0, \quad x \geq 0 \Big\}.$$

Background information

The optimal value gives an upper bound for the maximal density of a ***sphere packing*** in n-dimensional Euclidean space. It stems from Cohn and Elkies [1].



Optimal planar sphere packing.

It is conjectured that this upper bound matches the known lower bound in dimensions $n = 1, 2, 8, 24$. This has only been proved for $n = 1$. For dimensions $n = 8, 24$ the question of finding an optimal sphere packing is open although there are no doubts that the root lattice E_8 and the Leech lattice Λ_{24} give *the* optimal sphere packing in their dimension. The case $n = 2$ can be solved by e.g. by different, elementary methods.

Cohn and Elkies gave a numerical heuristic which for $n = 24$ gives 30 digits of precision (see Cohn and Kumar [2]) but this computation required several weeks of computation. I don't think that the linear program above behaves numerically very good, but I also don't know how to improve on it. A related infinite dimensional linear program was studied in [3].

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12 Around the Burnside Problem

Problem proposed by Pierre-Emmanuel Caprace (IHES)

Statement of Problems

A *torsion group* is a group all of whose elements have finite order. A group is said to have *bounded exponent* if there exists an integer n such that the order of every element divides n. The following questions are central problems in group theory:

Is every finitely presented torsion group finite?

Is every finitely presented group of bounded exponent finite?

It is possible to formulate numerous variations on these questions in various contexts. The following non-discrete version was proposed by George Willis:

Let G be a compactly generated topological group. Assume that every element normalizes some compact open subgroup. Does G have a compact open normal subgroup?

The following questions lies between the discrete and the non-discrete case:

Let G be a topological group and $\Gamma < G$ be a discrete subgroup. Assume that every element of Γ normalizes some compact open subgroup. Is there a compact open subgroup which is globally normalized by Γ?

Background information

The above questions relate to the *general Burnside problem*, posed by William Burnside in 1902, which is one of the oldest and most influential questions in group theory. It asks whether a finitely generated torsion group is necessarily finite. A negative answer to this question was provided by Golod–Shafarevich in 1964, while the first examples of finitely generated infinite groups of bounded exponent were constructed in 1968 by Adian–Novikov. In 1994, E. Zelmanov was awarded the Fields medal for solving the so-called *restricted Burnside problem*, showing that *a finitely generated residually finite group of bounded exponent is necessarily finite* (a group is called *residually finite* if the intersection of all finite index subgroups is trivial). It is utterly intriguing that even the general Burnside problem remains open for *finitely presented* groups.

For a short history and detailed references on the Burnside problem, see [hist]. For a detailed account on Zelmanov's solution to the restricted Burnisde problem, see [VL]. Finally, see is [OS] for a more advanced reading on the construction of an infinite finitely presented torsion-by-cyclic group.

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