1. Classical analogies between numbers and functions

<table>
<thead>
<tr>
<th>Functions</th>
<th>Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{C}[x]$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>finite extensions $K/\mathbb{C}(x)$</td>
<td>finite extensions $K/\mathbb{Q}$</td>
</tr>
<tr>
<td>compact Riemann surface $S$</td>
<td>$\text{Spec } \mathcal{O}_K$</td>
</tr>
<tr>
<td>points on $S$</td>
<td>places of $K$</td>
</tr>
<tr>
<td>Jacobian variety $Cl^0(S)$</td>
<td>divisor class group $Cl(K)$</td>
</tr>
<tr>
<td>unramified finite abelian covers $S' \to S$ and link with $Cl^0(S)$ via $H_1(S, \mathbb{Z})$</td>
<td>unramified finite abelian extensions $K'/K$ and link with $Cl(K)$ (Hilbert class field)</td>
</tr>
<tr>
<td>ramified version of this</td>
<td>class field theory, ray class groups</td>
</tr>
<tr>
<td>non-abelian version of this $(\pi_1(S \setminus {P_1, ..., P_n}))$</td>
<td>non-abelian class field theory ($G(K^a/K)$ and Langlands)</td>
</tr>
<tr>
<td>Riemann-Roch</td>
<td>arithmetic analogue of Riemann-Roch</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Differential calculus and differential equations on $S$</th>
<th>NO CLASSICAL ANSWER FOR A POSSIBLE ANSWER SEE NEXT SECTIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{C}[x_1, x_2]$</td>
<td>NO CLASSICAL ANSWER MAYBE $\mathbb{Z} \otimes_{\mathbb{F}_q} \mathbb{Z}$ FOR A HYPOETHETICAL $\mathbb{F}_q$???) FOR AN ALTERNATIVE NEXT SECTIONS</td>
</tr>
</tbody>
</table>
2. ODEs, Part I

<table>
<thead>
<tr>
<th>Ordinary diff equations satisfied by functions</th>
<th>Ordinary “diff” equations satisfied by numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>(differential algebra)</td>
<td>(arithmetic differential algebra)</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Main reference for material below</th>
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<tr>
<th>Applications</th>
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<tbody>
<tr>
<td>Geometric Lang Conjecture (AB, Annals 92)</td>
<td>Arithmetic Th of the kernel (Invent 95)</td>
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<tr>
<td>Effective bound for GLC (AB, Duke 93)</td>
<td>Effective Manin-Mumford (AB, Duke 96)</td>
</tr>
<tr>
<td>Effective geo Manin-Mumford (AB, Duke 94)</td>
<td>Modular forms (AB, Crelle 2000)</td>
</tr>
<tr>
<td></td>
<td>Heegner points (AB+Poonen, Compositio 2009)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>R a ring</th>
<th>R a ring, ( p \in \mathbb{Z} ) a prime, non-zero divisor in ( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta : R \to R ) is a derivation if</td>
<td>( \delta : R \to R ) is a ( p )-derivation if ( \delta 1 = 0 ) and</td>
</tr>
<tr>
<td>( \delta(x + y) = \delta x + \delta y )</td>
<td>( \delta(x + y) = \delta x + \delta y - \frac{(x+y)^p - x^p - y^p}{p} )</td>
</tr>
<tr>
<td>( \delta(xy) = x\delta y + y\delta x )</td>
<td>( \delta(xy) = x^p\delta y + y^p\delta x + p\delta x\delta y )</td>
</tr>
<tr>
<td>(Ritt, Kolchin)</td>
<td>(Joyal C.R Acad Sci Canada 85, AB Invent 95)</td>
</tr>
</tbody>
</table>

for theory below: AB Invent 95, Duke 96
<table>
<thead>
<tr>
<th>$\delta$ is a derivation iff</th>
<th>$\delta$ is a $p$-derivation iff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R \to D_2(R) = (R \times R$, dual number structure)</td>
<td>$R \to W_2(R) = (R \times R$, Witt vector structure)</td>
</tr>
<tr>
<td>$x \mapsto (x, \delta x)$ is a ring homomorphism</td>
<td>$x \mapsto (x, \delta x)$ is a ring homomorphism.</td>
</tr>
</tbody>
</table>

$R^\delta = \{ x \in R; \delta x = 0 \}$ subring of $R$

$R^\delta = \{ x \in R; \delta x = 0 \}$ submonoid of $R$

### Examples

1) $R = \mathbb{C}[x], \delta = d/dx$
   
   $R^\delta = \mathbb{C}$

2) $R = C^\infty(N, \mathbb{C}), N = \mathbb{R}, \mathbb{R}/\mathbb{Z}$
   
   $\delta = d/dx$

   $R^\delta = \mathbb{C}$

$\delta$ is a $p$-derivation iff $\phi : R \to R$

$\phi(x) = x^p + p\delta x$ is a ring homomorphism

so $p$-derivations $R \to R$ are in bijection with ring homomorphisms $\phi : R \to R$ lifting Frobenius $R/pR \to R/pR$.

$R^\delta = \{ -1, 0, 1 \}$

$R^\delta = \{ x \in R; x^p = x, f \geq 1 \}$

$R = \{ \sum_{i \geq 0} c_i p^i; c_i \in R^\delta \}$.  

Notation: $\hat{\cdot}$ means $p$-adic completion

### $f : R^N \to R$ is a $\delta$-function of order $\leq n$

if there exists $P \in R[T, T', ..., T^{(n)}]$ \((T, T', ... \ N\text{-tuples of variables})\)

s.t. $f(u) = Pu := P(u, \delta u, ..., \delta^nu), u \in R^N$

### $f : R^N \to \hat{R}$ is a $\delta$-function of order $\leq n$

if there exists $P \in R[T, T', ..., T^{(n)}]$ \((T, T', ... \ N\text{-tuples of variables})\)

s.t. $f(u) = Pu := P(u, \delta u, ..., \delta^nu), u \in R^N$
Globalizing: $X$ smooth scheme over $R$

$X(R) = \{ R - \text{points of } X \}$

(If $R = C^\infty(N, \mathbb{C})$ $X$ is the analogue of a submersion $M \to N$ of $C^\infty$ manifolds and $X(R)$ is the analogues of the set $C^\infty_N(N, M)$ of sections of $M \to N$)

$\to J^n(X) \to J^{n-1}(X) \to \cdots \to J^0(X) = \hat{X}$

projective system of schemes (jet spaces)

for $X$ affine

$X = \text{Spec } R[T]/(f)$, with $T, f$ tuples

$J^n(X) = \text{Spec } R[T, T', ..., T^{(n)}]/(f, \delta f, ..., \delta^n f)$

$B_n := R[T, T', ..., T^{(n)}], B = \bigcup B_n$

$\delta : B \to B$

$\delta$ unique derivation lifting $\delta$ on $R$

such that $\delta T = T'$, $\delta T' = T''$, ...

for $X$ non-affine: glue

for $X$ non-affine: glue

$O(J^n(X)) \to \text{Map}(X(R), R)$

$[\Phi(x, x', ...)] \mapsto (\alpha \mapsto \Phi(\alpha, \delta \alpha, ...))$

$O^n(X) = \text{image}$

elements called $\delta$-functions on $X$ of order $\leq n$

(intuitively: algebraic diff equations)

$O(J^n(X)) \to \text{Map}(\hat{X}(R), \hat{R})$

same

$O^n(X) = \text{image}$

elements called $\delta$-functions on $X$ of order $\leq n$

(intuitively: arithmetic diff equations)
\[
\mathcal{O}^{n-1}(X) \subset \mathcal{O}^n(X)
\]
\[
\delta : \mathcal{O}^{n-1}(X) \to \mathcal{O}^n(X)
\]
total differential operator
(Cartan distribution)

\[
\mathcal{O}^{n-1}(X) \subset \mathcal{O}^n(X)
\]
\[
\delta : \mathcal{O}^{n-1}(X) \to \mathcal{O}^n(X)
\]
arithmetic total differential operator

From now on, for convenience, \( R \) is a \( \delta \)-closed field (i.e. \( \text{char} \ 0 \) and for any \( f, g \in R[T, T', T'', \ldots] \) with \( T \) one variable and \( \text{ord}(g) < \text{ord}(f) \) there exists \( u \in R \) with \( f(u, \delta u, \ldots) = 0, \ g(u, \delta u, \ldots) \neq 0 \) Then \( \mathcal{O}(J^n(X)) = \mathcal{O}^n(X) \)

\[
\mathcal{O}^{n-1}(X) \subset \mathcal{O}^n(X)
\]
From now on \( R = \widehat{\mathbb{Z}}_{ur} \)

\[
\mathcal{O}^{n-1}(X) \subset \mathcal{O}^n(X)
\]
Then \( \mathcal{O}(J^n(X)) = \mathcal{O}^n(X) \)

Main problems

1) Compute \( \mathcal{O}^n(X) \) for concrete \( X \)
2) Compute \( \mathcal{O}^n(X)^\Gamma \) for a correspondence \( \Gamma \to X \times X \)
(classical differential invariant theory)
gives algebro-geometric structure on \( X/\Gamma \)
compare with A. Connes

\[
J^1(X) \neq T(X), \ J^n(X) \neq \text{arc spaces}
\]
unless \( X \) descends to \( R^d \)

\[
J^1(X) \neq T(X), \ J^n(X) \neq \text{arc spaces}
\]
unless \( X \) descends to \( R^d \)

\[
J^n(X) \to J^{n-1}(X)
\]
is a torsor
for a vector bundle (hence a Zariski locally trivial fibration with fiber an affine space \( \hat{\mathbb{A}}^d \))

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J^n(X) \to J^{n-1}(X)
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is a torsor
for a vector bundle (hence a Zariski locally trivial fibration with fiber an affine space \( \hat{\mathbb{A}}^d \))

\[
X = G \text{ group implies } J^n(G) \text{ groups}
\]
\[
N^n := \text{Ker}(J^n(G) \to G)
\]
For \( G \) commutative
\[
N^n \cong \mathbb{G}^{nd}_a \text{ (as groups)}
\]
\[
X = G \text{ group implies } J^n(G) \text{ groups}
\]
\[
N^n := \text{Ker}(J^n(G) \to \widehat{G})
\]
For \( G \) commutative
\[
N^n \cong \hat{\mathbb{G}}^{nd}_a \text{ (as formal schemes)}
\]
\[
N^n \cong \hat{\mathbb{G}}^{nd}_a \text{ as groups, in general.}
\]
For \( G = \mathbb{G}_a, \mathbb{G}_m, E \) (\( E \) elliptic curve)
\[
N^1 \cong \hat{\mathbb{G}}_a \text{ (as groups)}
\]
and there exists a homo \( \chi : N^n \to \mathbb{G}^n_a \)
\[
\chi = (\chi_1, \ldots, \chi_n), \ \chi_1, \ldots, \chi_n \text{ linearly independent.}
<table>
<thead>
<tr>
<th>Theorem 1 (AB, AmerJM 93). $\mathcal{O}^n(\mathbb{P}^N) = R.$</th>
<th>Theorem 1 (AB, Duke 96). $\mathcal{O}^n(\mathbb{P}^N) = R.$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Theorem 2</strong> (Fuchs-Manin). For $X$ elliptic curve there exists a $\delta$-function $\phi : X(R) \to R$ of order $\leq 2$ which is a non-zero homomorphism. If $X$ defined over $R_0 \subset R$, $\delta R_0 \subset R_0$, $\text{tr.deg.} R_0 / R_0^\delta = 1$ then $(\text{Ker } \psi) \cap X(R_0) = X(R_0)^{\text{tors}}$</td>
<td><strong>Theorem 2</strong> (AB, Invent 95) For $X$ elliptic curve there exists a $\delta$-function $\phi : X(R) \to R$ of order $\leq 2$ which is a non-zero homomorphism. Moreover $[\text{Ker } \psi : p^\infty X(R)] &lt; \infty$.</td>
</tr>
<tr>
<td><strong>Proof.</strong> First part similar to $\to$ cf. AB $\neq$ Manin.</td>
<td><strong>Proof.</strong></td>
</tr>
<tr>
<td>$0 \to N^2 \to J^2(X) \to \hat{X} \to 0$</td>
<td></td>
</tr>
<tr>
<td>$\text{Hom}(J^2(X), \hat{G}_a) \to \text{Hom}(N^2, \hat{G}_a) \to H^1(X, \mathcal{O})$</td>
<td></td>
</tr>
<tr>
<td>there exist $a_1, a_2 \in R$, $\partial(a_1\chi_1 + a_2\chi_2) = 0$, etc.</td>
<td>similar statement for abelian schemes</td>
</tr>
<tr>
<td>similar statement for abelian schemes</td>
<td></td>
</tr>
<tr>
<td><strong>Theorem 3</strong> (Manin). $X$ as in Theorem 2. There exists $\psi$ of order 1 iff $X$ descends to $R^\delta$.</td>
<td><strong>Theorem 3</strong> (AB, Invent 95). $X$ as in Theorem 2. There exists $\psi$ of order 1iff $X$ is a canonical lift (CL).</td>
</tr>
<tr>
<td><strong>Theorem 4</strong> (AB, AmerJM 94). $X$ projective curve of genus $\geq 2$ that does not descend to $R^\delta$. Then $\mathcal{O}^1(X)$ separates the points of $X(R)$. In fact $J^n(X)$ are affine for $n \geq 1$.</td>
<td><strong>Theorem 4</strong> (AB, Duke 96). $X$ projective curve of genus $\geq 2$. Then $\mathcal{O}^1(X)$ separates the points of $X(R)$. In fact $J^n(X)$ are affine for $n \geq 1$.</td>
</tr>
<tr>
<td>Theorem 4 embeds a projective curve into an affine space via $\delta$-functions: $X(R) \to R^N$</td>
<td>Theorem 4 implies Manin-Mumford with effective bound, see below.</td>
</tr>
<tr>
<td>--------------------------</td>
<td>--------------------------</td>
</tr>
<tr>
<td>$X/\mathbb{C}$ a curve of genus $g \geq 2$ in an abelian variety $A$, $X$ not def over $\mathbb{Q}$. Then $\sharp(X \cap A_{tors}) \leq C(g)$</td>
<td>$X/\mathbb{C}$ a curve of genus $g \geq 2$ in its Jacobian $A$, $X$ def over $\mathbb{Q}$. Then $\sharp(X \cap A_{tors}) \leq C(g,p)$ where $p$ smallest prime of good reduction</td>
</tr>
<tr>
<td>(finiteness conjectured by Manin-Mumford, and proved by Raynaud) $C(g)$ conjectured by Mazur)</td>
<td>(finiteness conjectured by Manin-Mumford and proved by Raynaud $C(g)$ conjectured by Mazur)</td>
</tr>
<tr>
<td>Proof Heavily uses $\delta$. Skipped.</td>
<td>Proof. May replace $\mathbb{C}$ by $R$. Enough $\sharp(X(R) \cap A(R)_{\text{prime-to-$p$-tors}}) \leq C(g,p)$ Any point $P$ in this set lifts to an $R$-point $J^1(P)$ of $J^1(X) \subset J^1(A)$ The reduction mod $p$ of $J^1(P)$ of $J^1(A)$ lies in $J^1(X) \cap pJ^1(A)$. This intersection is finite because it’s affine intersect projective Cardinality bounded by Bezout.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem 6 (AB, AmerJM 95).</th>
<th>OPEN</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$ as in Theorem 4. Then the $\delta$-functions $X(R) \subset \text{Jac}(X)(R) \xrightarrow{\psi} R$ generate the field $\text{Frac}(\mathcal{O}^1(X))$ over $R$ (but not necessarily the $R$-algebra $\mathcal{O}^1(X)$.)</td>
<td></td>
</tr>
</tbody>
</table>
δ-modular forms: 
theory less rich but 
still interesting 
(cf. eg. Ramanujan, 
AB, Crelle 2000.)

\[ X \subset X_1(N) \text{ open, disjoint from cusps} \]
\[ X(R) \subset \{(A, \alpha); A/R \text{ an elliptic curve, } \alpha \text{ a level } \Gamma_1(N) \text{ structure}\} \]
\[ E \xrightarrow{\pi} X \text{ universal elliptic curve} \]
\[ L := \pi_* \Omega_{E/X} \]
\[ V := \text{Spec} \left( \bigoplus_{n \geq 0} L^\otimes n \right) \]
\[ V^* := V \backslash O \text{ a } \mathbb{G}_m\text{-torsor} \]
\[ V^*(R) \subset \{(A, \alpha, \omega); \omega \text{ a 1-form on } A\} \]

\[ \delta\text{-modular function of order } \leq n: \text{ any element of } M^n := \mathcal{O}^n(V^*) \text{ viewed as a map } V^*(R) \rightarrow R \]
\[ M^\infty := \bigcup_n M^n \]

\[ W := \mathbb{Z}[\phi] = \{ \sum a_i \phi^i; a_i \in \mathbb{Z} \} \]
\[ \text{deg}: W \rightarrow \mathbb{Z}, \text{ deg}(\sum a_i \phi^i) = \sum a_i \]

\[ \delta\text{-modular form of weight } w \in W: \text{ any } f \in M^n, \ f : V^*(R) \rightarrow R \text{ such that } f(\lambda \cdot a) = \lambda^w f(a), \ \lambda \in R^\times, a \in V^*(R) \]
\[ M^n(w) = \{ \delta\text{-modular forms of weight } w \text{ and order } \leq n \}. \]
Let $w \in W$ with $\text{deg}(w)$ even.

$f \in M^n(w)$ is isogeny covariant if for any

$(A_1, \alpha_1, \omega_1), (A_2, \alpha_2, \omega_2)$

representing $P_1, P_2 \in V^n(R)$

and any isogeny $u : A_1 \to A_2$ of degree prime to $p$

with $u^*\omega_2 = \omega_1$

we have $f(P_1) = (\text{deg}(u))^{-\text{deg}(w)/2} f(P_2)$.

$I^n(w) := \{ f \in M^n(w) ; f \text{ isogeny covariant} \},$

$I(w) = I^{\text{ord}(w)}(w)$

NB. $\text{Proj} (\bigoplus_w I(w))$

is “morally” $\frac{X_1(N)}{\text{Hecke correspondences}}$

Aim of the theory: to compute $M^n(w), I^n(w), M^\infty, ...$

Fourier (expansion) map $M^\infty \to \hat{\mathbb{R}}((q))$

analogue of $f^1$

(related to Ramanujan)

but no analogue of $f^\partial$

**Theorem 7** (AB+Barcau+Saha).

Assume $X$ is in the ordinary locus.

Then $\bigoplus_w I(w)$ $\delta$-generated by 2 forms

$f^1 \in M^1(-1 - \phi), f^\partial \in M^1(\phi - 1)$

Moreover:

1) The kernel of the Fourier map is $\delta$-generated by $f^1$ and $f^\partial - 1$

2) The $p$-adic closure of the image of the Fourier map is Katz’s ring

$\mathcal{W}$ of generalized $p$-adic modular forms.

moral: the divided congruences of Katz can all be obtained from

arithmetic differential objects.
$$y^2 = x^3 + a_4 x + a_6,$$

$a_4, a_6$ indeterminates

$$\Delta = \Delta(a_4, a_6)$$

the discriminant

$$\mathbb{C}[a_4, a_6, \Delta^{-1}] \to \mathcal{O}(V^*)$$

$$\mathbb{C}[a_4, a_6, a'_4, a'_6, \Delta^{-1}] \to \mathcal{O}^1(V^*)$$

$$a'_4 \mapsto q \frac{d}{dq} (a_4(q)) \in \mathbb{C}((q)) \text{ via Fourier}$$

$$a'_4 \mapsto \frac{a_4(q^p) - a_4(q)}{p} \in \hat{\mathbb{R}}((q)) \text{ via Fourier}$$
\[
f^1 = \frac{2a_4a'_6 - 3a_6a'_4}{\Delta}
\]

\(f^1 \mapsto 1\) via Fourier by Ramanujan

\(f^1 \mapsto 0\) via Fourier

\(f_0\) related to Kronecker’s modular polynomial mod \(p^2\)

<table>
<thead>
<tr>
<th>Jet construction of (f^1)</th>
<th>Jet construction of (f^1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(via usual Kodaira-Spencer)</td>
<td>universal elliptic curve (E = \bigcup U_i \to \text{Spec } M^\infty)</td>
</tr>
<tr>
<td>(s_i : \widehat{U}_i \to J^1(U_i)) sections of the natural projection</td>
<td></td>
</tr>
<tr>
<td>(s_i - s_j : \widehat{U}_i \cap \widehat{U}_j \to N^1 \simeq \widehat{G}_a)</td>
<td></td>
</tr>
<tr>
<td>get class (\eta \in H^1(\widehat{E}, \mathcal{O}) = H^1(E, \mathcal{O}))</td>
<td></td>
</tr>
<tr>
<td>(f^1 = \text{Serre dual of } \eta \cup \frac{dx}{y}).</td>
<td></td>
</tr>
</tbody>
</table>

Note \(f^1(P) = 0\) iff \(P\) has CL

Construction of \(f^\partial\) (Barcau, Compositio 2002)

\[
f^\partial = \text{const} \cdot (72(a_6^p + pa_6')\frac{\partial}{\partial a_6} - 16(a_4^p + pa_4')^2\frac{\partial}{\partial a_6} - p(P^p + p\delta P))(f^1)
\]

\(P\) the Ramanujan form viewed as \(p\)-adic modular form, \(P \in R[a_4, a_6, \Delta^{-1}, E_{p-1}^{-1}]\)

both \(f^1\) and \(f^\partial\) admit crystalline constructions

**Theorem 8** (AB+Poonen, Duke 2009)

\(S\) modular curve, \(A\) elliptic curve, \(X\) curve over \(\mathbb{C}\), \(\pi: X \to S, \varphi: X \to A\), \(CM \subset S\) CM locus, \(\Gamma \subset A\) subgroup

\(\text{rank}(\Gamma) := \dim_{\mathbb{Q}}(\Gamma \otimes \mathbb{Q}) < \infty\). Then \(\sharp(\pi^{-1}(CM) \cap \varphi^{-1}(\Gamma)) < \infty\).
No interesting analogue

Theorem 9 (AB+Poonen, Compositio 2009)

$S, A, X, \pi, \varphi$ as above and over $R = \mathbb{Z}_p^\text{ur}$

$CL \subset S(R)$ CL locus. Then there exists a constant $c$
such that for any subgroup $\Gamma \subset A(R)$

$\#(\pi^{-1}(CL) \cap \varphi^{-1}(\Gamma)) < c \cdot p^{\text{rank}(\Gamma)}$

Proof. (Case $\text{rank}(\Gamma) = 0, \varphi = \text{id}$)

AB, Invent 95 gives homo $\psi : A(R) \to R$ of order 2

Let $f^\sharp = \psi \circ \varphi : S(R) \to R$

AB, Crelle 2000 gives $f^\flat : S^\dagger(R) \to R$

(constructed from $f^1$ above, $S^\dagger \subset S$)

vanishing on $CL$. So any $P$ in intersection

is a solution of the system $f^\sharp = f^\flat = 0$

Claim: there are $h_0, h_1 : S^\dagger(R) \to R$ such that

$f^\sharp - h_0 f^\flat - h_1 \delta f^\flat$

has order 0. (I.e. one can

eliminate the derivatives in the system of
differential equations to get an equation

without derivatives.) The latter has only

finitely many zeros (Strassman)

and $P$ is one of them.
### 4. PDEs: Hyperbolic and Parabolic Type

<table>
<thead>
<tr>
<th>Fix $R = \mathbb{C}^\infty(\mathbb{R}_x, \mathbb{C})$ with coordinate $x$</th>
<th>Fix $R = \mathbb{Z}_p^{nr}$ with “coordinate” $p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fix $A = \mathbb{C}^\infty(\mathbb{R}_x \times \mathbb{R}_t, \mathbb{C})$ with coordinates $x, t$</td>
<td>$A = R[[q]]$ with “coordinates” $p, q$</td>
</tr>
<tr>
<td>$(A, \partial_x, \partial_t)$</td>
<td>$(A, \delta_p, \delta_q)$</td>
</tr>
<tr>
<td>$\partial_x u = \frac{\partial u}{\partial x}$</td>
<td>$\delta_p u = \frac{u^{(\omega)(q^r)} - u(q)^p}{p}$</td>
</tr>
<tr>
<td>$\partial_t u = \frac{\partial u}{\partial t}$</td>
<td>$\delta_q u = q\frac{\partial u}{\partial q}$</td>
</tr>
<tr>
<td>$P : A \rightarrow A$</td>
<td>$P : A \rightarrow A$</td>
</tr>
<tr>
<td>$Pu = P(t, x, u, ..., \partial_x^i \partial_t^j u, ...)$</td>
<td>$Pu = P(u, ..., \delta_p^i \delta_q^j u, ...)$</td>
</tr>
<tr>
<td>$P$ polynomial in $u$ and the partials more generally same for manifolds</td>
<td>$P$ a $p$-adic limit of polynomials more generally $P : X(A) \rightarrow A$ for $X$ smooth scheme over $A$.</td>
</tr>
</tbody>
</table>

$P = \psi$ called linear with constant coefficients if

$\psi u = \sum_{i,j} c_{ij} \partial_x^i \partial_t^j u$

$c_{ij} \in \mathbb{C}$

Symbol: $\sigma_P = \sum_{i,j} c_{ij} \sqrt{-1} \xi^i \tau^j$

$\psi = \psi_r$ means $\psi$ has order $r$

$\psi = \psi_x$ means $\psi$ involves $\partial_x$ only same with $t$

$\psi = \psi_xt$ means $\psi$ involves both $\partial_x$ and $\partial_t$

For $X = G$ group $P = \psi$ called linear if

$\psi : G(A) \rightarrow A$ a homomorphism

(and same for any extension of $A$ on which $\delta$s operate with appropriate comm rel)

there is an analogue

$\psi = \psi_r$ means $\psi$ has order $r$

$\psi = \psi_p$ means $\psi$ involves $\delta_p$ only same with $q$

$\psi = \psi_{pq}$ means $\psi$ involves both $\delta_p$ and $\delta_q$

Concerned with $G = \mathbb{G}_a, \mathbb{G}_m, E$

sometimes write $\psi_a, \psi_m, \psi_E$
Examples of $P$-classification

1. **Theorem 10 (AB+Simanca)**
   
   \[ \frac{\partial_t u}{\partial x} (\text{convection}) \]
   
   1) For $G_a$ all $\psi$s are built from $Id$ via $\phi_p$ and $\delta_q$
   
   2) For $G_m$ all $\psi$s built from \[ \psi_p^1 u = \frac{\delta u}{\partial x} - \frac{u}{2} \left( \frac{\delta u}{\partial x} \right)^2 + \ldots \]
   
   3) For $G_m$ all $\psi$s built from \[ \psi_q^1 u = \frac{\delta u}{\partial x} \]
   
   4) For $E/A$ general:
   
   a) No analogues of $\psi_p^1, \psi_q^1$
   
   b) SURPRISE!!! There is a $\psi_{pq}^1$ (convection eq)
   
   c) There is a $\psi_{p}^2$ (Manin 63) and a $\psi_{q}^2$ (AB, Invent 95)
   
   d) All $\psi$s are built from the above
   
   e) one relation $\psi_q^2 + \lambda \psi_p^2 = \psi_a^1 \circ \psi_{pq}^1$
   
   (canonical wave eq; $\lambda$ unique, interesting invariant)
   
   4) For $E/R$ general
   
   a) No analogues of $\psi_p^1, \psi_q^1$
   
   b) SURPRISE!!! There is a $\psi_{pq}^1$ (convection eq)
   
   c) There is a $\psi_{p}^2$ (Manin 63) and a $\psi_{q}^2$ (AB, Invent 95)
   
   d) All $\psi$s are built from the above
   
   e) one relation $\psi_q^2 + \lambda \psi_p^2 = \psi_a^1 \circ \psi_{pq}^1$
   
   (canonical wave eq; $\lambda$ unique, interesting invariant)

solution space $U = \{u \in A; \psi u = 0\}$

stationary solutions:

\[ \partial_t u = 0 \]

\[ \psi_q u = 0 \] (i.e. $u \in G(R)$)

(there is also a good definition for case 3)

no analogue, problem trivial

Theorem 11 (AB+Simanca)

Complete classification of $\psi$s that admit non-stationary solutions

(Morally a quantization phenomenon:

e.g. heat equation in 4 above has non-stationary solutions iff $\lambda$ is in the set of all $Z$-multiples of a certain element in $R$.)

Convolution.

the “ring” $R_*$ is the group $R$

with “multiplication”

\[(f \ast g)(x) = \int f(y)g(x - y)dy\]

$A_*$ is the group $A$

viewed as $R_*$-“module” under $\ast$

$U$ is a $R_*$-“submodule”

“” because convolution not always defined; also need distributions....
**Fundamental solutions**

For $P = \psi$ with constant coefficients
If $\mathcal{U} \subset A \to R^p$
$u \mapsto (u, \partial_t u, ...) |_{t=0}$
is a bijection
(boundary value well posed)
then $\mathcal{U}$ is a free $R_\ast$-“module”
under $\ast$ of rank $\rho$
(basis of fundamental solutions
is a basis mapped to Dirac $\times$ identity matrix)

**Theorem 12 (AB+Simanca)**

Let $\psi : G(A) \to A$ be non-degenerate
(a condition on the symbol $\sigma(\xi, \tau u)$)
Let $\mathcal{U}_1$ be the group of solutions
vanishing at $q = 0$
Then convolution module structure of $\mathcal{U}_1$
descends to an $R$-module structure and
$\mathcal{U}_1$ is a finitely generated free $R$-module
Its rank is the number of positive integer
roots of $\sigma(0, \tau)$

**Exponential solutions**

$Pu = 0$ gives by Fourier inversion in $x$
$\sigma_P(\xi, -\sqrt{-1}\tau) \hat{u}(\xi, t) = 0$
onordinary diff eqn with parameter $\xi$
its solutions are linear combinations of exponentials
again by Fourier inversion
$u(x, t) = \sum_{j=1}^\rho \int C_j(\xi) e^{-\sqrt{-1}\xi x - \sqrt{-1}\tau_j(\xi) t} d\xi$
$\tau_j(\xi)$ roots of $\sigma_P(-\xi, -\tau)$.

Above can be also viewed as an analogue of
exponential solutions
5. PDEs: elliptic type

| $A = C^\infty(D, \mathbb{C})$, $D \subset \mathbb{C}$, $z, \bar{z} \in A$ | $A = \mathbb{Z}, \mathbb{Z}[\zeta_m, 1/N]$, $p_1, p_2 \in A$ |
| $\partial_z, \partial_{\bar{z}} : A \to A$ | $\delta_{p_1}, \delta_{p_2} : A \to A$ |
| $\delta_{p_i} a = \frac{\phi_{p_i}(a) - a^{p_i}}{p_i}$ |

| $M \to D$ a $C^\infty$ submersion | $X \to \text{Spec } A$ smooth scheme |
| $M(A) := C^\infty_D(D, M)$ set of sections | $X(A)$ set of sections i.e. of $A$-points |

| $\psi : M(A) \to R$ (non linear diff operators) | $\psi : X(A) \to R$ (non linear diff operators) |
| $u \mapsto \psi u = P(z, \bar{z}, u, ..., \partial_z^i \partial_{\bar{z}}^j u, ...)$ | $u \mapsto \psi u = P(u, ..., \delta_{p_1} \delta_{p_2}^i u, ...)$ |
| $P$ a polynomial in $u$ and the partials | $P$ a polynomial |
| $B$ the ring of all such operators | $B$ the ring of all such operators |
| need to allow variable $A$ | (need to allow variable $A$) |
| If $\psi$ has order $r$ write $\psi = \psi^r$ | If $\psi$ has order $r$ write $\psi = \psi^r$ |
| If $\psi$ only involves $z$ write $\psi = \psi_z$ | If $\psi$ only involves $p_1$ write $\psi_{p_1}$ |

| If $M = G$ family of Lie groups over $D$ and $\psi$ homo, $\psi$ called linear | If $X = G$ is a group scheme over $A$ and $\psi$ homo we COULD call $\psi$ a linear arithmetic differential operator |
| PROBLEM: $B$ does not contain such non-zero $\psi$s in most cases | SO WE NEED ANOTHER DEFINITION |
| FACT: In many cases for any affine $X = \text{Spec } B \subset G$ the completions $\hat{B}_{p_1}$ and $\hat{B}_{p_2}$ in the $p_1$-adic and $p_2$-adic topologies contain non-zero “linear elements” $\psi_{p_1}$ and $\psi_{p_2}$ respectively. | |
| AIM: to “analytically continue” $\psi_{p_1}$ and $\psi_{p_2}$ DILEMMA: $\psi_{p_1}$ and $\psi_{p_2}$ are defined on disjoint spaces $X^{\hat{p}_1}$ and $X^{\hat{p}_2}$ |
No problem here

Main idea: assume for simplicity \( A = \mathbb{Z}[1/N] \)
and set \( A_0 = \mathbb{Z}_{(p_1)} \cap \mathbb{Z}_{(p_2)} \),
\( B_0 = B \otimes_A A_0 \)
say \( \psi_{p_1} \) and \( \psi_{p_2} \)
can be analytically continued along a section \( P \in X(A) \)
(with ideal \( I \) in \( B \))
if there exists \( \psi_0 \in B_0 \tilde{I} \)
which coincides with \( \psi_{p_1} \) and \( \psi_{p_2} \)
in the rings \( B_0^{(I, p_1)} \) and \( B_0^{(I, p_2)} \)
respectively. (PICTURE!!!!)

one may assume for all practical purposes that
\( B_0^{I} = A_0[[t]] \), \( t \) a tuple
and what we require is that there is an element in \( A_0[[t]] \)
whose images in \( \mathbb{Z}_{p_1}[[t]] \) and \( \mathbb{Z}_{p_2}[[t]] \)
coincide with the images of \( \psi_{p_1} \) and \( \psi_{p_2} \)

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DEFINITION: a linear arithmetic partial differential operator
is a pair \( \psi = (\psi_{p_1}, \psi_{p_2}) \) of linear elements as above that
can be analytically continued along the zero section of \( G \)
(Write \( \psi : G \to \mathbb{G}_a \).)

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Example

\( G = \mathbb{C}^\times \times D \to D \)
\( G(R) = C^\infty(D, \mathbb{C}^\times) \)
\( \mathbb{G}_a(R) = R = C^\infty(D, \mathbb{C}) \)
\( \psi^2 : C^\infty(D, \mathbb{C}^\times) \to C^\infty(D, \mathbb{C}) \)
\( \psi^2 : \mathbb{G}_m \to \mathbb{G}_a \)

\( \psi^2 u = \frac{1}{2} \Delta \log u = \partial_x \partial_x u \)
\( \psi^2_{p_1} u = \left( 1 - \frac{\phi_{p_2}}{p_2} \right) \sum (-1)^{n+1} p_1^n \left( \frac{\delta_{p_1} u}{u + T} \right)^n \)
\( \psi^2_{p_2} = \left( 1 - \frac{\phi_{p_1}}{p_1} \right) \sum (-1)^{n+1} p_2^n \left( \frac{\delta_{p_2} u}{u + T} \right)^n \)

\( \psi^2_{p_1} \in \mathbb{Z}_{p_1} [x, x^{-1}, \delta_{p_1} x, \delta_{p_2} x, \delta_{p_1} \delta_{p_2} x]^{[p_1]} \)
\( \psi^2_{p_2} \in \mathbb{Z}_{p_2} [x, x^{-1}, \delta_{p_1} x, \delta_{p_2} x, \delta_{p_1} \delta_{p_2} x]^{[p_2]} \)

They can be analytically continued because they come from the same series in
\( A_0[[T, \delta_{p_1} T, \delta_{p_2} T, \delta_{p_1} \delta_{p_2} T]] \)
via \( x \mapsto T + 1 \).
\[
\psi^2 u = \partial_z \left( \frac{\partial u}{\partial z} \right) = \partial_{\xi} \left( \frac{\partial u}{\partial \eta} \right)
\]

(“Dirac decomposition”)

the analytic continuation above

is an analogue of the “Dirac decomposition”

| Theorem 13 (AB+Simanca, Advances Math 2009). |
| All linear arithmetic partial differential operators on \( \mathbb{G}_m \) are obtained from \( \psi^2 \) above. |

| Example |
| universal elliptic curve over \( D \) |
| \( E = (D \times \mathbb{C})/\sim \rightarrow D \) |
| \( \psi^4 : C^\infty_D(D,E) \rightarrow C^\infty(D,\mathbb{C}) \) |
| \( \psi^4 u = \frac{1}{16} \Delta \Delta \log_E u = \partial_z^2 \partial_{\xi}^2 \log_E u \) |
| where \( \log_E : E \rightarrow D \times \mathbb{C} \rightarrow \mathbb{C} \) |
| is the multivalued logarithm (again, Dirac decomposition) |

| Another view on analytic continuation between primes: cf. Borger+AB |

| Theorem 14 (AB+Simanca, Advances Math 2009). |
| If \( E \) has ordinary reduction at \( p_1, p_2 \) then all linear arithmetic partial differential operators on \( E \) are obtained from \( \psi^4 \) above. |