A Dynamic Logic of Knowledge and Access

Eric Pacuit

Center for Logic and Philosophy of Science
Tilburg University
ai.stanford.edu/~epacuit

(Joint work with Tomohiro Hoshi)

February 26, 2010
A recurring issue in any formal model representing agents’ (changing) informational attitudes is how to account for the fact that the agents are limited in their access to the available inference steps, possible observations and available messages.
A recurring issue in any formal model representing agents’ (changing) informational attitudes is how to account for the fact that the agents are limited in their access to the available inference steps, possible observations and available messages.

1. This may be because the agents are not logically omniscient and so do not have unlimited reasoning ability.
A recurring issue in any formal model representing agents’ (changing) informational attitudes is how to account for the fact that the agents are limited in their access to the available inference steps, possible observations and available messages.

1. This may be because the agents are not logically omniscient and so do not have unlimited reasoning ability.

2. But it can also be because the agents are following a predefined protocol that explicitly limits statements available for observation and/or communication.
Logical Omniscience

\[ [K_i(\varphi \rightarrow \psi) \land K_i \varphi] \rightarrow K_i \psi \]
Logical Omniscience

\[ [K_i(\varphi \to \psi) \land K_i\varphi] \to K_i\psi \]

\[ \vdash \varphi \implies \vdash K_i\varphi \]
Logical Omniscience

\[ [K_i(\varphi \rightarrow \psi) \land K_i \varphi] \rightarrow K_i \psi \]

\[ \vdash \varphi \quad \implies \quad \vdash K_i \varphi \]

\[ \vdash \varphi \leftrightarrow \psi \quad \implies \quad \vdash K_i \varphi \leftrightarrow K_i \psi \]
Access Sets

Epistemic model: $\mathcal{M} = \langle W, \{\sim i\}_{i \in A}, V \rangle$

- $W$ a nonempty set of states
- $\sim i \subseteq W \times W$ encodes $i$’s (hard) information
- $V$ is a valuation function assigning sets of formulas to atomic propositions: $V : \text{At} \rightarrow \wp(W)$
Access Sets

Epistemic model: \( \mathcal{M} = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, V \rangle \)

- \( W \) a nonempty set of states
- \( \sim_i \subseteq W \times W \) encodes \( i \)'s (hard) information
- \( V \) is a valuation function assigning sets of formulas to atomic propositions: \( V : \text{At} \rightarrow \wp(W) \)

Epistemic model with access sets: \( \mathcal{M} = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, V, \{A_i\}_{i \in \mathcal{A}} \rangle \)

where \( A_i : W \rightarrow \wp(\mathcal{L}) \) with \( \mathcal{L} \) a formal language: \( A_i(w) \) is the set of formulas that \( i \) currently has access to in state \( w \).
What is the access set?

An agent’s access set...
What is the access set?

An agent’s access set...

1. is the (current) \textit{outcome} of the information gathering process. 

\textit{typically as a result of a logical inference step or observation, but other (trusted) agents may contribute to the access set through communication.}
What is the access set?

An agent’s access set...

1. is the (current) \textit{outcome} of the information gathering process. 

   \textit{typically as a result of a logical inference step or observation, but other (trusted) agents may contribute to the access set through communication.}

2. \textit{constrains} the current information gathering process. 

   \textit{i.e., the sentences that an agent can (is permitted to, has the ability to) observe or infer.}
Questions

1. How should we extend the basic modal language to reason about epistemic models with access sets?

2. What do agents “know” about the other agents’ access sets?

3. What dynamic operations change the access sets over time?
Reasoning with Protocols: An Example

1. Uma is a physician whose neighbour is ill. Uma does not know and has not been informed. Uma has no obligation (as yet) to treat the neighbour.

2. Uma is a physician whose neighbour Sam is ill. The neighbour’s daughter Ann comes to Uma’s house and tells her. Now Uma does have an obligation to treat Sam, or perhaps call in an ambulance or a specialist.

3. Mary is a patient in St. Gibson’s hospital. Mary is having a heart attack. The caveat which applied in case 1. does not apply here. The hospital has an obligation to be aware of Mary’s condition at all times and to provide emergency treatment as appropriate.

Reasoning with Protocols: An Example

1. Uma is a physician whose neighbour is ill. Uma does not know and has not been informed. Uma has no obligation (as yet) to treat the neighbour.

2. Uma is a physician whose neighbour Sam is ill. The neighbour’s daughter Ann comes to Uma’s house and tells her. Now Uma does have an obligation to treat Sam, or perhaps call in an ambulance or a specialist.

3. Mary is a patient in St. Gibson’s hospital. Mary is having a heart attack. The caveat which applied in case 1. does not apply here. The hospital has an obligation to be aware of Mary’s condition at all times and to provide emergency treatment as appropriate.

Reasoning with Protocols: An Example

1. Uma is a physician whose neighbour is ill. Uma does not know and has not been informed. Uma has no obligation (as yet) to treat the neighbour.

2. Uma is a physician whose neighbour Sam is ill. The neighbour’s daughter Ann comes to Uma’s house and tells her. Now Uma does have an obligation to treat Sam, or perhaps call in an ambulance or a specialist.

3. Mary is a patient in St. Gibson’s hospital. Mary is having a heart attack. The caveat which applied in case 1. does not apply here. The hospital has an obligation to be aware of Mary’s condition at all times and to provide emergency treatment as appropriate.

Reasoning with Protocols: An Example

1. Uma is a physician whose neighbour is ill. Uma does not know and has not been informed. Uma has no obligation (as yet) to treat the neighbour.

2. Uma is a physician whose neighbour Sam is ill. The neighbour’s daughter Ann comes to Uma’s house and tells her. Now Uma does have an obligation to treat Sam, or perhaps call in an ambulance or a specialist.

3. Mary is a patient in St. Gibson’s hospital. Mary is having a heart attack. The caveat which applied in case 1. does not apply here. The hospital has an obligation to be aware of Mary’s condition at all times and to provide emergency treatment as appropriate.

Example 1 & 2

$t = 0$

$t = 1$

$t = 2$

$t = 3$
Example 1 & 2

\[ t = 0 \]

\[ t = 1 \]

\[ t = 2 \]

\[ t = 3 \]
Example 1 & 2

\[ t = 0 \]

\[ t = 1 \]

\[ t = 2 \]

\[ t = 3 \]

\[ \lambda_u(v) = \lambda_u(c) \]
Example 1 & 2

$t = 0$

$m$
$c$
$t$

$t = 1$

$m$
$c$
$t$
$m$
$c$
$t$

$t = 2$

$t$
$c$
$t$
$c$
$t$
$c$
$t$

$t = 3$

Eric Pacuit
Example 1 & 2

\[ \lambda_u(vm) = \lambda_u(cm) \]
Example 2

\[ \lambda_u(\nu m) = \lambda_u(cm) \]
Example 2

$H, 2 \models K_u G(t)$

Eric Pacuit
Ann has the (knowledge based) obligation to tell Uma about her father’s illness ($K_a G(m)$).
Ann has the (knowledge based) obligation to tell Uma about her father’s illness ($K_a G(m)$).

Clearly, Ann will not be under any obligation to tell Uma that her father is ill, if Ann justifiably believes that Uma would not treat her father even if she knew of his illness.
Ann has the (knowledge based) obligation to tell Uma about her father’s illness ($K_a G(m)$).

Clearly, Ann will not be under any obligation to tell Uma that her father is ill, if Ann justifiably believes that Uma would not treat her father even if she knew of his illness.

Thus, to carry out a deduction we will need to assume

$$K_a(K_u \text{sick} \leftrightarrow \Diamond \text{treat})$$
A similar assumption is needed to derive that Uma has an obligation to treat Sam.
A similar assumption is needed to derive that Uma has an obligation to treat Sam.

Obviously, if Uma has a good reason to believe that Ann always lies about her father being ill, then she is under no obligation to treat Sam.
A similar assumption is needed to derive that Uma has an obligation to treat Sam.

Obviously, if Uma has a good reason to believe that Ann always lies about her father being ill, then she is under no obligation to treat Sam.

In other words, we need to assume

$$K_u(msg \leftrightarrow \text{sick})$$
Common Knowledge of Ethicality

These formulas can all be derived for one common assumption which we call *Common Knowledge of Ethicality*. 
Common Knowledge of Ethicality

These formulas can all be derived for one common assumption which we call *Common Knowledge of Ethicality*.

1. The agents must (commonly) know the protocol.
2. The agents are all of the same “type” (social utility maximizers)
What is a Protocol?

Given the full tree $T$ of events, a protocol is any subtree of $T$. 
What is a Protocol?

Given the full tree $T$ of events, a protocol is any subtree of $T$.

▶ A protocol is the set of histories compatible with some process, i.e., it is the “unwinding” of a (multi-agent) state transition system.
What is a Protocol?

Given the full tree $T$ of events, a protocol is any subtree of $T$.

- A protocol is the set of histories compatible with some process, i.e., it is the “unwinding” of a (multi-agent) state transition system.

- A protocol is the set of histories satisfying some property:
  - Physical properties: every message is eventually answered, no message is received before it is sent
  - Agent types: agent $i$ is the type of agent who always lies, agent $j$ is the type who always tells the truth
What is a Protocol?

Given the full tree $T$ of events, a **protocol** is any subtree of $T$.

- A protocol is the set of histories compatible with some **process**, i.e., it is the “unwinding” of a (multi-agent) state transition system.

- A protocol is the set of histories satisfying some **property**: 
  - Physical properties: every message is eventually answered, no message is received before it is sent
  - Agent types: agent $i$ is the **type** of agent who always lies, agent $j$ is the type who always tells the truth

- A protocol is the set of histories of an extensive game consistent with a (partial) **strategy profile**.
Epistemic Dynamics and Protocols

ETL methodology: when describing a social situation, first write down all possible sequences of events, then at each moment write down the agents' uncertainty, from that infer how the agents' knowledge changes from one moment to the next.

DEL methodology: describe an initial situation, provide a method for how events change a model that can be described in the formal language, then construct the event tree as needed.
Epistemic Dynamics and Protocols

**ETL methodology:** when describing a social situation, first write down all possible sequences of events, then at each moment write down the agents’ uncertainty, from that infer how the agents’ knowledge changes from one moment to the next.
Epistemic Dynamics and Protocols

**ETL methodology:** when describing a social situation, first write down all possible sequences of events, then at each moment write down the agents’ uncertainty, from that infer how the agents’ knowledge changes from one moment to the next.

**DEL methodology:** describe an initial situations, provide a method for how events change a model that can be described in the formal language, then construct the event tree as needed.
Background: DEL

Ann would like Bob to attend her talk; however, she only wants Bob to attend if he is interested in the subject of her talk, not because he is just being polite.

There is a very simple procedure to solve Ann’s problem: have a (trusted) friend tell Bob the time and subject of her talk.

Is this procedure correct?
Background: DEL

Ann would like Bob to attend her talk; however, she only wants Bob to attend if he is interested in the subject of her talk, not because he is just being polite.

There is a very simple procedure to solve Ann’s problem: have a (trusted) friend tell Bob the time and subject of her talk.

Is this procedure correct? Yes, if

1. Ann knows about the talk.
2. Bob knows about the talk.
3. Ann knows that Bob knows about the talk.
4. Bob does not know that Ann knows that he knows about the talk.
5. And nothing else.
Background: DEL

$P$ means “The talk is at 2PM”.
Background: DEL

$P$ means “The talk is at 2PM”.

$\mathcal{M}, s \models K_A P \land \neg K_B P$
Background: DEL

$P$ means “The talk is at 2PM”.

$\mathcal{M}, s \models K_A P \land \neg K_B P$
Background: DEL
Background: DEL

\[(\mathcal{M} \otimes E_1) \otimes E_2\]
Background: DEL

The initial model (Ann and Bob are ignorant about $P_{2PM}$).

Private announcement to Ann about the talk.
Background: An Abstract Description of the Communication Event

The event Charles tells Bob that the talk is at 2PM.
Background: An Abstract Description of the Communication Event

The event Charles tells Bob that the talk is at 2PM.
The event Charles tells Bob that the talk is at 2PM.

Ann knows which event took place.
Background: An Abstract Description of the Communication Event

The event Charles tells Bob that the talk is at 2PM.

Bob thinks a different event took place.
Background: An Abstract Description of the Communication Event

The event Charles tells Bob that the talk is at 2PM.

That is, Bob learns the time of the talk, but Ann learns nothing.
Background: Product Update
Background: Product Update
Background: Product Update

\[ P, B, P^1 \]

\[ \neg P, t \]

\[ (s, e_1) \quad P \]

\[ P \quad (s, e_2) \]

\[ (s, e_3) \quad P \]

\[ \neg P \quad (t, e_3) \]
Background: Product Update

\[ (s, e_1) \bigcirc P \]

\[ (s, e_2) \]

\[ (s, e_3) \]

\[ \neg P \]
Background: Product Update

\[
(s, e_1) \models \neg K_B K_A K_B P \\
(s, e_1) \quad P \\
(s, e_3) \quad P \\
\neg P \quad (t, e_3)
\]
Background: Product Update

\[(s, e_1) \models \neg K_B K_A K_B P\]
Background: Product Update

\[(s, e_1) \models \neg K_B K_A K_B P\]

\[(s, e_1) \quad P \quad (s, e_2)\]

\[(s, e_3) \quad P \quad \neg P \quad (t, e_3)\]
Background: Product Update

\((s, e_1) \models K_B K_A K_B P\) \hspace{1cm} \((s, e_1) \rightarrow P \leftarrow (s, e_3)\) \hspace{1cm} \((s, e_2) \rightarrow P \leftarrow (t, e_3)\)
Background: Product Update Details

Let $\mathcal{M} = \langle W, \sim, V \rangle$ be an epistemic model.

An event model is a tuple $\mathcal{E} = \langle E, \rightarrow, \text{Pre} \rangle$, where $\rightarrow \subseteq E \times E$ and $\text{Pre} : E \rightarrow \mathcal{L}$. 
Background: Product Update Details

Let $\mathcal{M} = \langle W, \sim, V \rangle$ be an epistemic model.

An event model is a tuple $\mathcal{E} = \langle E, \rightarrow, \text{Pre} \rangle$, where $\rightarrow \subseteq E \times E$ and $\text{Pre} : E \rightarrow \mathcal{L}$.

The update model $\mathcal{M} \otimes \mathcal{E} = \langle W', \sim', V' \rangle$ where
Let $\mathcal{M} = \langle W, \sim, V \rangle$ be an epistemic model.

An **event model** is a tuple $\mathcal{E} = \langle E, \rightarrow, \text{Pre} \rangle$, where $\rightarrow \subseteq E \times E$ and $\text{Pre} : E \rightarrow \mathcal{L}$.

The **update model** $\mathcal{M} \otimes \mathcal{E} = \langle W', \sim', V' \rangle$ where

- $W' = \{(w, a) \mid w \models \text{Pre}(a)\}$
Background: Product Update Details

Let $\mathcal{M} = \langle W, \sim, V \rangle$ be an epistemic model.

An event model is a tuple $\mathcal{E} = \langle E, \rightarrow, Pre \rangle$, where $\rightarrow \subseteq E \times E$ and $Pre : E \to \mathcal{L}$.

The update model $\mathcal{M} \otimes \mathcal{E} = \langle W', \sim', V' \rangle$ where

- $W' = \{(w, a) \mid w \models Pre(a)\}$
- $(w, e) \sim' (w', e')$ iff $w \sim w'$ and $e \rightarrow e'$
Background: Product Update Details

Let $\mathcal{M} = \langle W, \sim, V \rangle$ be an epistemic model.

An event model is a tuple $\mathcal{E} = \langle E, \rightarrow, Pre \rangle$, where $\rightarrow \subseteq E \times E$ and $Pre : E \rightarrow \mathcal{L}$.

The update model $\mathcal{M} \otimes \mathcal{E} = \langle W', \sim', V' \rangle$ where

- $W' = \{ (w, a) \mid w \models Pre(a) \}$
- $(w, e) \sim' (w', e')$ iff $w \sim w'$ and $e \rightarrow e'$
- $(w, e) \in V(p)$ iff $w \in V(p)$
Background: Product Update Details

Let $\mathcal{M} = \langle W, \sim, V \rangle$ be an epistemic model.

An event model is a tuple $\mathcal{E} = \langle E, \rightarrow, Pre \rangle$, where $\rightarrow \subseteq E \times E$ and $Pre : E \rightarrow \mathcal{L}$.

The update model $\mathcal{M} \otimes \mathcal{E} = \langle W', \sim', V' \rangle$ where

1. $W' = \{(w, a) \mid w \models Pre(a)\}$
2. $(w, e) \sim' (w', e')$ iff $w \sim w'$ and $e \rightarrow e'$
3. $(w, e) \in V(p)$ iff $w \in V(p)$

$\mathcal{M}, w \models [\mathcal{E}, e]\varphi$ iff $\mathcal{M}, w \models Pre(e)$ implies $\mathcal{M} \otimes \mathcal{E}, (w, e) \models \varphi$. 

Eric Pacuit
Background: Literature


**Observation:** By repeatedly updating an epistemic model with event models, the machinery of DEL creates ETL models.
**Observation:** By repeatedly updating an epistemic model with event models, the machinery of DEL creates ETL models.

Let $\mathcal{M}$ be an epistemic model, and $P$ a DEL protocol (tree of event models). The ETL model generated by $\mathcal{M}$ and $P$, $\text{Forest}(\mathcal{M}, P)$, represents all possible evolutions of the system obtained by updating $\mathcal{M}$ with sequences from $P$.

If $\mathcal{X}$ is a set of DEL protocols, we define

$$F(\mathcal{X}) = \{ \text{Forest}(\mathcal{M}, P) \mid \mathcal{M} \text{ an epistemic model and } P \in \mathcal{X} \}$$
Example: Constrained Public Announcements

\[ P, Q \rightarrow Q, R \]

\[ P, Q, R \rightarrow Q, R \]

\[ P, Q \rightarrow Q, R \]

\[ P, Q, R \rightarrow P, R \]

\[ !P \rightarrow Q \rightarrow R \]
Example: Constrained Public Announcements

\[ s \quad P, Q \quad \overset{i}{\leftarrow} \quad Q, R \quad \overset{v}{\rightarrow} \quad P, R \]
\[ t \quad P, Q, R \quad \overset{j}{\rightarrow} \quad P, R \]

\[ (s, P, Q) \quad (t, P, Q) \quad (t, P, R) \quad (s, P, R) \quad (v, Q, R) \quad (u, P, R) \]

\[ !P \quad !Q \quad !R \]
Example: Constrained Public Announcements
Example: Constrained Public Announcement

\[ (s) \xrightarrow{(t)} (u) \xrightarrow{(v)} \]

\[ (s, !P) \xrightarrow{(t, !P)} (u, !P) \]

\[ (s, !P, !Q) \xrightarrow{(t, !P, !Q)} (u, !P, !R) \]
Example: Constrained Public Announcement
Example: Constrained Public Announcement

\[
\begin{align*}
(s, P, Q) & \rightarrow (t, P, !P) \\
(s, !P, !Q) & \rightarrow (t, !P, !Q) \\
(t, !P, !Q) & \rightarrow (u, !P, !Q) \\
\end{align*}
\]
Example: Constrained Public Announcement
Example: Constrained Public Announcement

\[ (s) \xrightarrow{!P} (t) \xrightarrow{!P} (u) \xrightarrow{!P} (v) \]

\[ (s, !P) \xrightarrow{!P} (t, !P) \xrightarrow{!P} (u, !P) \]

\[ (s, !P, !Q) \xrightarrow{!Q} (t, !P, !Q) \xrightarrow{!Q} (u, !P, !Q) \]

\[ (t, !P, !R) \xrightarrow{!R} (u, !P, !R) \]
Example: Constrained Public Announcement

\[
\begin{align*}
(s) & \rightarrow (t) & (u) & \rightarrow (v) \\
(s, !P) & \rightarrow (t, !P) & (u, !P) & \rightarrow (v) \\
(s, !P, !Q) & \rightarrow (t, !P, !Q) & (t, !P, !R) & \rightarrow (u, !P, !R) \\
\end{align*}
\]
Example: Constrained Public Announcement

\[(t) \models R \land \neg \langle !R \rangle \top\]

\[
(s) \rightarrow (t) \rightarrow (u) \rightarrow (v)
\]

\[
(s, !P) \rightarrow (t, !P) \rightarrow (u, !P)
\]

\[
(s, !P, !Q) \rightarrow (t, !P, !Q) \rightarrow (u, !P, !R)
\]

\[
!P \quad !P \quad !P
\]

\[
!Q \quad !Q \quad !R
\]
\[ F(X) = \{ \text{Forest}(\mathcal{M}, P) \mid \mathcal{M} \text{ an epistemic model and } P \in X \}. \]

- Can we characterize the class of ETL models \( F(X) \)?

- Can we axiomatize interesting classes of DEL-generated ETL models?

A Characterization Theorem

Let $\Sigma$ be a finite set of events and suppose $X^{uni}_{DEL}$ is the class of uniform DEL protocols (with a finiteness condition).

**Characterization Theorem** A model is in $\mathbb{F}(X^{uni}_{DEL})$ iff it satisfies propositional stability, synchronicity, perfect recall, local no miracles, and local bisimulation invariance.
Constrained Public Announcement

1. $A \rightarrow \langle A \rangle^T$ vs. $\langle A \rangle^T \rightarrow A$

Let $X_{PAl}$ be the set of all PAl protocols.

Theorem. There is a sound and complete axiomatization of $F(X_{PAl})$.
Constrained Public Announcement

1. \( A \rightarrow \langle A \rangle^T \) vs. \( \langle A \rangle^T \rightarrow A \)

2. \( \langle A \rangle K_i P \leftrightarrow A \land K_i \langle A \rangle P \)
Constrained Public Announcement

1. $A \rightarrow \langle A \rangle^T$ vs. $\langle A \rangle^T \rightarrow A$

2. $\langle A \rangle K_i P \leftrightarrow A \land K_i \langle A \rangle P$

3. $\langle A \rangle K_i P \leftrightarrow \langle A \rangle^T \land K_i (A \rightarrow \langle A \rangle P)$
Constrained Public Announcement

1. $A \rightarrow \langle A \rangle \top \text{ vs. } \langle A \rangle \top \rightarrow A$

2. $\langle A \rangle K_i P \leftrightarrow A \land K_i \langle A \rangle P$

3. $\langle A \rangle K_i P \leftrightarrow \langle A \rangle \top \land K_i (A \rightarrow \langle A \rangle P)$

4. $\langle A \rangle K_i P \leftrightarrow \langle A \rangle \top \land K_i (\langle A \rangle \top \rightarrow \langle A \rangle P)$
Constrained Public Announcement

1. \( A \rightarrow \langle A \rangle \top \) vs. \( \langle A \rangle \top \rightarrow A \)

2. \( \langle A \rangle K_i P \leftrightarrow A \land K_i \langle A \rangle P \)

3. \( \langle A \rangle K_i P \leftrightarrow \langle A \rangle \top \land K_i (A \rightarrow \langle A \rangle P) \)

4. \( \langle A \rangle K_i P \leftrightarrow \langle A \rangle \top \land K_i (\langle A \rangle \top \rightarrow \langle A \rangle P) \)

Let \( X_{PAL} \) be the set of all PAL protocols.

**Theorem.** There is a sound and complete axiomatization of \( \mathbb{F}(X_{PAL}) \).
Questions

1. A public announcement is one specific type of event model, can we axiomatize classes of ETL models generated by other types of event models?

2. Which formal languages are best suited to describe these DEL generated ETL models?
Semi-Private Announcement

\[ A \xrightarrow{G} P \xleftarrow{e} f \]
Language

\[ \varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid K_i \varphi \mid \langle \varphi \rangle \varphi \mid A_i \varphi \]
Language

$$\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid Ki\varphi \mid \langle \varphi \rangle \varphi \mid Ai\varphi$$

- $Ki\varphi$ is intended to mean “according to $i$’s current information $\varphi$ is true”

- $\langle \varphi \rangle \psi$ is intended to mean “after $\varphi$ is made publicly available, $\psi$ is true”

- $Ai\varphi$ is “agent $i$ has access to $\varphi$” (“agent $i$ can observe $\varphi$” or “agent $i$ has the ability to observe $\varphi$”).
Definition. A protocol is a function \( p : A \times \mathbb{N} \rightarrow \wp(\mathcal{L}) \) such that, for every \( n \in \mathbb{N} \), \( i \in A \), and \( \varphi \in \mathcal{L} \), \( \varphi \in p(i, n) \) iff \( \neg \varphi \in p(i, n) \).

We denote the set of protocols by \( Ptcl \).
Making $\psi$ public

Let $P = (W, \{\sim_i\}_{i \in A}, V, p)$ be an epistemic model with a protocol.

Incorporating the observation of $\psi$ into $P$ is the model $P \otimes \psi = (W', \{\sim'_i\}_{i \in A}, V', p')$ where

\[
\begin{align*}
W' & := W \\
\sim'_i & := \begin{cases} 
\sim_i & \text{if } \psi \not\in p(i, 0) \\
\{(w, v) \in \sim_i | P, w \models \psi \iff P, v \models \psi\} & \text{if } \psi \in p(i, 0)
\end{cases} \\
p'(i, n) & := p(i, n + 1)
\end{align*}
\]
\[ \mathcal{P}, w \models p \quad \text{iff} \quad w \in V(p) \quad (\text{with } p \in P) \]
\[ \mathcal{P}, w \models \neg \varphi \quad \text{iff} \quad \mathcal{P}, w \not\models \varphi \]
\[ \mathcal{P}, w \models \varphi \land \psi \quad \text{iff} \quad \mathcal{P}, w \models \varphi \quad \text{and} \quad \mathcal{P}, w \models \psi \]
\[ \mathcal{P}, w \models K_i \varphi \quad \text{iff} \quad \forall v \in W : \text{ if } w \sim_i v \text{ then } \mathcal{P}, v \models \varphi \]
\[ \mathcal{P}, w \models \langle \psi \rangle \varphi \quad \text{iff} \quad \mathcal{P}, w \models \psi \quad \text{and} \quad \mathcal{P} \otimes \psi, w \models \varphi \]
\[ \mathcal{P}, w \models A_i \varphi \quad \text{iff} \quad \varphi \in p(i, 0) \]
1. \( \langle \alpha \rangle \langle \beta \rangle \varphi \rightarrow \langle \langle \alpha \rangle \beta \rangle \varphi \)

2. \([p]K_i p\) where \(p \in \text{At}\)

3. \(\langle \theta \rangle K_i \varphi \leftrightarrow \langle \theta \rangle \top \land K_i(\langle \theta \rangle \top \rightarrow \langle \theta \rangle \varphi)\)
Axioms

R1 $\langle \theta \rangle p \leftrightarrow \theta \land p$ where $p \in \text{At}$

R2 $\langle \theta \rangle \neg \varphi \leftrightarrow \theta \land \neg \langle \theta \rangle \varphi$

R3 $\langle \theta \rangle (\varphi \land \psi) \leftrightarrow \langle \theta \rangle \varphi \land \langle \theta \rangle \psi$

R4 $\langle \theta \rangle K_i \varphi \leftrightarrow \theta \land (A_i \theta \rightarrow K_i [\theta] \varphi) \land$
$\neg A_i \theta \rightarrow K_i ([\theta] \varphi \land [\bar{\theta}] \varphi)$
Axioms

R1 \( \langle \theta \rangle p \leftrightarrow \theta \land p \) where \( p \in At \)

R2 \( \langle \theta \rangle \neg \varphi \leftrightarrow \theta \land \neg \langle \theta \rangle \varphi \)

R3 \( \langle \theta \rangle (\varphi \land \psi) \leftrightarrow \langle \theta \rangle \varphi \land \langle \theta \rangle \psi \)

R4 \( \langle \theta \rangle K_i \varphi \leftrightarrow \theta \land (A_i \theta \rightarrow K_i[\theta] \varphi) \land \\
(\neg A_i \theta \rightarrow K_i([\theta] \varphi \land [\bar{\theta}] \varphi)) \)

P-neg \( A_i \varphi \leftrightarrow A_i \bar{\varphi} \)

Ptcl \( A_j \varphi \rightarrow K_i A_j \varphi \)

Uni \( \langle \alpha \rangle A_i \varphi \rightarrow [\beta] A_i \varphi \)
Describing Protocols

\[ \text{ptcl-A} \quad A_i \varphi \land A_i \psi \rightarrow A_i (\varphi \land \psi) \]

\[ \text{ptcl-K} \quad A_i (\varphi \rightarrow \psi) \rightarrow (A_i \varphi \rightarrow A_i \psi). \]

\[ \text{Ref} \quad A_i \varphi \rightarrow A_i A_i \varphi \]

\[ \text{F-Mon} \quad A_i \varphi \rightarrow [\alpha] A_i \varphi \]

\[ \text{P-Mon} \quad [\alpha] A_i \varphi \rightarrow A_i \varphi \]

\[ \text{Exp} \quad K_i \varphi \rightarrow A_i \varphi \]

Theorem(s). (T. Hoshi and EP) There is a sound and complete axiomatization of \( F(\mathcal{X} \mathcal{S}_{\text{Priv}}) \) with \( \mathcal{X} \mathcal{S}_{\text{Priv}} \) satisfying any of the above properties.
Describing Protocols

ptcl-A  \[ A_i\varphi \land A_i\psi \rightarrow A_i(\varphi \land \psi) \]

ptcl-K  \[ A_i(\varphi \rightarrow \psi) \rightarrow (A_i\varphi \rightarrow A_i\psi). \]

Ref  \[ A_i\varphi \rightarrow A_iA_i\varphi \]

F-Mon  \[ A_i\varphi \rightarrow [\alpha]A_i\varphi \]

P-Mon  \[ [\alpha]A_i\varphi \rightarrow A_i\varphi \]

Exp  \[ K_i\varphi \rightarrow A_i\varphi \]

**Theorem(s).** (T. Hoshi and EP) There is a sound and complete axiomatization of \( F(X_{SPriv}) \) with \( X_{SPriv} \) satisfying any of the above properties.
Conclusions

- $F(\text{PAL})$, $F(\text{DEL})$, $F(\text{X}_{\text{PAL}})$, $F(\text{X}_{\text{DEL}})$, $F(\text{X}_{\text{SPriv}})$, ... 
- Compare with justification logics
- Use DEL to generate extensive games with imperfect information and unawareness
Thank you!