On parameterizations of the Nordheim function

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Abstract

Several parameterizations of tabulated values of the Nordheim function $\vartheta(y)$ and its complementary function $t(y)$ are proposed. The function $\vartheta(y)$ plays essential role in description of field emission.

1. Under the influence of a strong electric field $F$ (at field strength above $10^7$ V/cm) the surface of solids and liquids emits electrons. The phenomenon, known now as electron field emission, was discovered in 1897 by Robert Wood, who found that a high voltage applied between a pointed cathode and an anode caused a current to flow \footnote{After considerable experimenting R. Wood "succeeded in finding a method of producing X-rays by what appears to be a new form of cathode discharge, which manifests itself as a blue arc between two minute balls of platinum in a very high vacuum" \cite{2}.}. Later, in 1928, Fowler and Nordheim \cite{2} proposed a theory of field emission from metals, based on the newly created quantum mechanics, as electrons tunneling through a potential barrier.

In the framework of the Fowler-Nordheim theory, the current density of field emission electrons can be written in the following form \cite{2,4-7}

$$J = A \frac{F^2}{\varphi} \exp \left\{ -B \frac{\varphi^{3/2}}{F} \vartheta(y) \right\}, \quad (1)$$

where $J$ is the current density in A/cm$^2$, $F$ is electric field at surface in V/cm, $\varphi$ is the work function in eV. The field-independent constants $A$ and $B$ and the variable $y$ are

$$A = \frac{e^3}{8\pi\hbar} = 1.5414 \cdot 10^{-6}, \quad B = \frac{8\pi \sqrt{2m}}{3e\hbar} = 6.8309 \cdot 10^{-7}, \quad y = \frac{\sqrt{e^3F}}{\varphi} = 3.7947 \cdot 10^{-4} \sqrt{\frac{F}{\varphi}} \quad (2)$$

where $-e$ is the charge on the electron, $m$ is the electron mass and $\hbar$ is Planck’s constant. Numerical coefficients in Eq.\,(2) corresponds to resent values of the physical constants \cite{8}.

The Nordheim function $\vartheta(y)$ takes into account a lowering of the potential barrier due to the image potential (the Shottky effect) and its distinction from an idealized triangular shape. The function $t(y)$ in the denominator of Eq.\,(1) is defined as

$$t(y) = \vartheta(y) - (2y/3)(d\vartheta/dy). \quad (3)$$

The expression of $\vartheta(y)$ (with a corrected definition of $k$ \cite{3}) is following

$$\vartheta(y) = \sqrt{\frac{1 + \sqrt{1 - y^2}}{2}} \cdot \left( E(k^2) - \frac{y^2K(k^2)}{1 + 2\sqrt{1 - y^2}} \right), \quad (4)$$

where

$$E(k^2) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \beta} \, d\beta, \quad K(k^2) = \int_0^{\frac{\pi}{2}} \frac{d\beta}{\sqrt{1 - k^2 \sin^2 \beta}}, \quad k^2 = \frac{2\sqrt{1 - y^2}}{1 + 2\sqrt{1 - y^2}}. \quad (5)$$

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$K(k^2)$ and $E(k^2)$ are the complete elliptic integrals of the first and second kind.

2. It follows from Eq. (4) that $\vartheta(y)$ has a complicated structure, and so for a long time only tabulated values of $\vartheta(y)$ and $t(y)$ [3]-[6] were used. From our view there is a need to have a convenient parameterization of $\vartheta(y)$ valid in whole range of $y \in [0, 1]$.

Figs. 1a and 1b shows the behavior of $\vartheta(y)$ and $t(y)$ with function values (points) taken from [3]-[6]. The function $\vartheta(y)$ varies significantly with $y$ (i.e. with the field variation), however $t(y)$ is quite close to unity at all values of $y$.

To fit $\vartheta(y)$ by elementary functions, several functional forms were tested. From them were chosen power functions Eq.(6), Eq.(7) and polynomial functions $P^{(n)}$ of different order $n=3÷8$, Eq.(8). The corresponding functions $t(y)$ were calculated with use of Eq.(3)

$$\vartheta(y) = p_0 + p_1 y^{p_2}, \quad t(y) = p_0 + p_1 (1 - \frac{2}{3} p_2) y^{p_2}, \quad (6)$$

$$\vartheta(y) = p_0 + p_1 y + p_2 y^{3/2}, \quad t(y) = p_0 + \frac{1}{3} p_1 \cdot y, \quad (7)$$

$$\vartheta(y) = P^{(n)}(y) = \sum_{i=0}^{n} p_i y^i, \quad t(y) = \sum_{i=0}^{n} (1 - \frac{2}{3} i) p_i \cdot y^i. \quad (8)$$

The fitted parameters $p_i$ are listed in the Table. The parameters were obtained with use of the MathCad program package [9]. Figs 1a, 1c and 2a demonstrate a very good quality of fit provided by (6)-(8). However, at large $y$ Eqs. (6)-(7) predict for $t(y)$ values slightly different (less 1%) from the tabulated values. Use of Eq.8 with $n=6, 8, \ldots$ allows us to eliminate this minor discrepancy (Fig. 2b).

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Figure 1: Functions $\vartheta(y)$ and $t(y)$ (points) parameterized (lines) by Eqs. (6), a), b) and Eqs. (7), c), d).
To conclude, the Nordheim elliptic function \( \vartheta(y) \) can be well fitted by functions of the type given by Eqs. (6)-(8).

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References


