Topological chaos and periodic braiding of almost-cyclic sets

Mark A. Stremler

Department of Engineering Science & Mechanics
Virginia Polytechnic Institute & State University


Physics of Mixing – 24 January 2011
Topological chaos through stirring

Complexity is ‘built in’ the flow due to the topology of boundary motions

Thurston-Nielsen classification theorem:

a periodic stirring motion can be one of three topological types:

- finite order – topologically trivial
- pseudo-Anosov – chaotic
- reducible – both f.o. & pA

Boyland, Aref & Stremler (2000) *JFM* 403, 277
Topological chaos in a viscous fluid

Move 3 rods on ‘figure-8’ paths through glycerin

• stirrers move on periodic orbits in two steps

• Thurston-Nielsen theorem gives a lower bound on stretching:
  \[ \lambda_{TN} = \frac{1}{2} (3 + \sqrt{5}) \]
  \[ h_{TN} = \log(\lambda_{TN}) = 0.962 \ldots \]

  non-trivial material lines grow like \( l \sim l_0 \lambda^n \)
  \[ \lambda \geq \lambda_{TN} \]
Topological chaos in a viscous fluid

finite order: chaos present due to fluid dynamics

finite order:  \( \lambda_{TN} = 1 \)
\( h_{TN} = \log(\lambda_{TN}) = 0 \)

pseudo-Anosov: chaos guaranteed by motion of stirrers

pseudo-Anosov:  \( \lambda_{TN} = \frac{1}{2}(3 + \sqrt{5}) \)
\( h_{TN} = \log(\lambda_{TN}) = 0.962 \ldots \)
‘Stirring’ with fluid particles

point vortices in a periodic domain

one rod moving on an epicyclic trajectory

Fluid is wrapped around ‘ghost rods’ in the fluid

\[ f.o. \text{ rod motion} + \text{ ghost rod motion} = \rho A \text{ ‘stirring’} \]
Model system: ‘Designing’ a flow with ghost rods

- design a flow with ghost rods that appear when and where we want them: a lid-driven cavity flow

\[
\psi = 0
\]

\[
\nabla^2 \nabla^2 \psi = 0
\]

\[
u = \frac{\partial \psi}{\partial y} = \pm \sum_{n=1}^{N} U_n \sin(nx/2)
\]

\[
u = \frac{\partial \psi}{\partial y} = \mp \sum_{n=1}^{N} U_n \sin(nx/2)
\]
Model system: Stokes flow in a lid-driven cavity

\[ u = \frac{\partial \psi}{\partial y} = \pm \sum_{n=1}^{N} U_n \sin(nx/2) \]

Exact solution for the stream function

\[ \psi(x, y) = U_1 \psi_1 + U_2 \psi_2 \]

\[ \psi_1 = C_1 f_1(y) \sin(x/2) \]

\[ \psi_2 = C_2 f_2(y) \sin(x) \]

\[ f_n(y) = y \cosh(nb/2) \sinh(ny/2) - b \sinh(nb/2) \cosh(ny/2) \]

\[ C_n = 2 \left[ \sinh(nb) + nb \right]^{-1} \]
generating pseudo-Anosov periodic orbits
\[ \psi = U \left[ \sqrt{1 - \beta} \psi_1 + \sqrt{\beta} \psi_2 \right] \]
\[ \beta = 0.414445 \ldots \quad U = 9.173958 \ldots \]

\[ x_L = (\pi - x_0, 0) \]
is a stagnation point

\[ x_C = (\pi, 0) \quad x_R = (\pi + x_0, 0) \]
change position after time \( \tau \)

\[ \{ x_L, x_C, x_R \} \]
trajectories produce a pseudo-Anosov braid
Aside: Generating these lid-driven flows

steady 3D flow in a rectangular channel

- surface grooves — Stroock et al. (2002) *Science*

- lid-driven secondary flow
  + Poiseuille flow
    - Chen & Stremler *Phys. Fluids*

- wide channel with surface grooves
Periodic points as ghost rods

elliptic points:  

hyperbolic points:  

parabolic points:  

parabolic points act *most* like flow around a solid rod  

*however* they are structurally unstable  

\[ \alpha = 1/3 \]
\[ \beta = 0.414445 \ldots \]
\[ U = 9.173958 \ldots \]

\[ h_{TN} = \ln \lambda_{TN} = 0.96242 \ldots \]
\[ h_{flow} = \ln \lambda \approx 0.964 \]

The TN theorem gives an excellent estimate of the flow behavior.
Perturbing the system

• change relative ‘strengths’ of $\psi_1$, $\psi_2$ (increase $\beta$)

$\beta = 0.415$

this ‘ghost rod’ structure acts like a semi-permeable rod
Perturbing $\beta : \psi = U \left[ \sqrt{1 - \beta} \psi_1 + \sqrt{\beta} \psi_2 \right]$
Perturbing $\beta : \psi = U \left[ \sqrt{1 - \beta} \psi_1 + \sqrt{\beta} \psi_2 \right]$
Perturbing $\beta : \psi = U \left[ \sqrt{1 - \beta} \psi_1 + \sqrt{\beta} \psi_2 \right]$
Perturbing the system: changing $\tau$

\[ \alpha = 1/3 \]
\[ \beta = 0.414445 \ldots \]
\[ U = 9.173958 \ldots \]

\[ \tau = \tau^* = 1.0 \]
Can we analyze stirring with ghost rods using topological chaos without having (or having to exactly find) low-order periodic points?

use a set-oriented approach and consider almost invariant or almost cyclic sets


almost invariant sets (AIS)
– regions of fluid that ‘stick together’ for a significant length of time

Probability is small that particles move between AIS in a short time
Set oriented approach: box formulation

- Bin the domain into a large number of boxes
  \[ \mathcal{B} = \{ B_1, \ldots, B_n \} \]

- The Transition Matrix or Ulam Matrix

\[
P_{ij} = \frac{\mu \left( B_i \cap f^{-1}(B_j) \right)}{\mu(B_i)}
\]

gives the transition probability from \( B_i \) to \( B_j \)

Ding, Li & Zhou (2002)
JCAM 147, 137
Set oriented approach: Almost Invariant Sets

- The *eigenvectors* of $P$ show the almost invariant sets (AIS)

- The (positive) *eigenvalues* of $P$ give a measure of how ‘leaky’ these sets are

  - Eigenvectors ordered by eigenvalue magnitudes
  - The first eigenvector has (largest) eigenvalue $= 1$
    - the domain itself is an invariant set

Calculations made by tracking a finite number of points
Set oriented approach: Almost Invariant Sets

- The *eigenvectors* of $P$ show the almost invariant sets (AIS)

- The (positive) *eigenvalues* of $P$ give a measure of how ‘leaky’ these sets are
  
  - Eigenvectors ordered by eigenvalue magnitudes
  
  - The first eigenvector has (largest) eigenvalue = 1
    - the domain itself is an invariant set
  
  - The (near) zero contour of the *second eigenvector* partitions the domain into two almost invariant sets

  - Lower eigenvectors give different partitions with higher transport rates = more “leaky”
Almost invariant sets in the ‘critical case’

almost invariant sets surround the periodic parabolic points
Almost invariant sets for $\tau < \tau^*$

Almost invariant sets persist even without periodic points.
Time dependent motion of AIS/ACS

AIS exhibit pA braiding motion = Almost Cyclic Sets (ACS)

• relative motion of the ACS “shadows” that of the (now non-existent) periodic orbits
Changes in ACS under perturbation

‘break-up’ of almost invariant sets appears to correspond to $h_{\text{flow}}$ passing below $h_{\text{TN}}$. 
• ACS reveal a collection of orbits that shadow periodic orbits of pA type for a finite number of periods

• the braid for the ACS is identical to the braid of the trajectories over a finite time (related to the leakiness of the ACS)

• the topological entropy for the ACS braid estimates the entropy for the flow

Stremler, Ross, Grover & Kumar, “Topological chaos and periodic braiding of almost-cyclic sets”, Physical Review Letters (in review)