Fibration categories and type theory

(Jeremy Avigad and) Chris Kapulkin

(Carnegie Mellon University and the) University of Pittsburgh

November 29th, 2011

Big question

How much homotopy theory can be described in type theory?

Equivalently, what homotopy-theoretic notions (i.e. invariant under weak equivalences) can we axiomatize/construct?

Examples

Homotopy-theoretic notions: homotopy fiber, being contractible, homotopy (co)limits.
Non-homotopy-theoretic notions: being a fibration.

Homotopy (co)limits

Example

The following diagrams are homotopy equivalent:

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \\ D^n & & \end{array} \qquad\qquad \begin{array}{ccc} S^{n-1} & \longrightarrow & \{*\} \\ \downarrow & & \\ \{*\} & & \end{array}$$

but their pushouts (S^n and $\{*\}$ respectively) are not.

Homotopy limits in type theory

Can we construct homotopy limits in type theory of Coq?

Framework

Models of homotopy theory

- model categories,
- quasicategories,
- (co)fibration categories,
- categories with weak equivalences,
- ...

Type theory seems to be best described in the framework of fibration categories.

Definition (Brown, 1973)

A category **C** with finite products is called a *fibration category*, if ...

Axiom 1.

$\mathbf{C}$ is equipped with two classes of maps: weak equivalences $\mathcal{W}$ and fibrations $\mathcal{F}$. Maps that belong to $\mathcal{W} \cap \mathcal{F}$ will be called trivial fibrations.

Definition

A is *contractible*, if

$$\sum_{a:A} \prod_{x:A} \mathrm{Paths}_A(x, a)$$

Definition

Given a map $f\colon B \longrightarrow A$ we define its *homotopy fiber* over $a : A$ to be the type

$$\mathrm{hfiber}(f, a) := \sum_{y:B} \mathrm{Paths}_A(fy, a).$$

Definition

A map $f \colon B \longrightarrow A$ is a *weak equivalence* if for all $y \colon B$, the homotopy fiber $\mathrm{hfiber}(f, y)$ is contractible.

Grad Students' Lemma

A map $f \colon B \longrightarrow A$ is a weak equivalence if and only if there exists some map $g \colon A \longrightarrow B$, inverse to f in that there are 'homotopies'

$$\varepsilon : \prod_{x:A} \mathrm{Paths}_A(fgx, x), \qquad \eta : \prod_{y:B} \mathrm{Paths}_B(y, gfy).$$

Definition

Type-theoretic fibrations are projections from dependent sums:

$$\sum_{x:A} B(x) \longrightarrow A$$

Axiom 2.

$\mathcal{W}$ satisfies 2-out-of-3 property (i.e. if any two of f, g, $g \circ f$ are weak equivalences, then so is the third).

This is an easy application of Grad Students' Lemma.

Axiom 3.

Both $\mathcal{F}$ and $\mathcal{W} \cap \mathcal{F}$ are stable under pullback.

Type-theoretically:

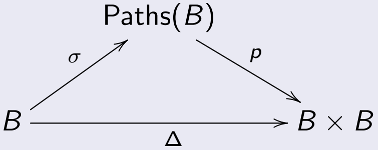
$$\begin{array}{ccc} \sum_{z:C} B(tz) & \longrightarrow & \sum_{x:A} B(x) \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ C & \xrightarrow[t]{} & A \end{array}$$

Lemma

For all $a : A$ we have $\mathrm{hfiber}(\pi_1, a) \simeq B(a)$.

Axiom 4.

For every object $B \in \mathbf{C}$ there exists a *path object* i.e. the exists a factorization of the diagonal map $\Delta = (1_B, 1_B)\colon B \longrightarrow B \times B$:



,

where σ is a weak equivalence and p is a fibration.

Take:

$$\mathrm{Paths}(B) = \sum_{x:B} \sum_{y:B} \mathrm{Paths}_B(x, y)$$

with $\sigma(x) = (x, x, \mathrm{refl}_B(x))$

Some properties and constructions

Factorization Lemma

Every maps $f\colon B \longrightarrow A$ admits a factorization $f = p \circ \sigma$, where $\sigma \in \mathcal{W}$ and $p \in \mathcal{F}$.

Right properness

The pullback of a weak equivalence along a fibration is again a weak equivalence.

Homotopy pullbacks

Given a diagram:

$$\begin{array}{ccc} & & A \\ & & \downarrow f \\ B & \xrightarrow[g]{} & C \end{array}$$

its homotopy pullback is defined as:

$$\sum_{x:A} \sum_{y:B} \mathrm{Paths}_C(fx, gy).$$

We also constructed some less trivial limits, however we are still working on a general construction of a homotopy limit.

Related work:

- Semantics of type theory in fibration categories (Arndt + K.).
- Model structure (given by HITs) as the basic framework to develop homotopy limits (Lumsdaine).
- Homotopy colimits in type theory + Univalence Axiom (Voevodsky).

15 minutes are probably over by now. . .