Motivation

Fibration categories and type theory

(Jeremy Avigad and) Chris Kapulkin

(Carnegie Mellon University and the) University of Pittsburgh

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**Big question**

How much homotopy theory can be described in type theory?

Equivalently, what homotopy-theoretic notions (i.e. invariant under weak equivalences) can we axiomatize/construct?

**Examples**

Homotopy-theoretic notions: homotopy fiber, being contractible, homotopy (co)limits.
Non-homotopy-theoretic notions: being a fibration.
Homotopy (co)limits

Example

The following diagrams are homotopy equivalent:

\[ S^{n-1} \rightarrow D^n \]
\[ S^{n-1} \rightarrow \{\ast\} \]

but their pushouts \((S^n \text{ and } \{\ast\} \text{ respectively})\) are not.

Homotopy limits in type theory

Can we construct homotopy limits in type theory of Coq?
Framework

Models of homotopy theory

- model categories,
- quasicategories,
- (co)fibration categories,
- categories with weak equivalences,
- ...

Type theory seems to be best described in the framework of fibration categories.

Definition (Brown, 1973)

A category $\mathbf{C}$ with finite products is called a fibration category, if ...

Axiom 1.

\(\mathcal{C}\) is equipped with two classes of maps: weak equivalences \(\mathcal{W}\) and fibrations \(\mathcal{F}\). Maps that belong to \(\mathcal{W} \cap \mathcal{F}\) will be called trivial fibrations.

Definition

A is contractible, if

\[
\sum \prod_{a:A \ x:A} \text{Paths}_A(x, a)
\]

Definition

Given a map \(f : B \rightarrow A\) we define its homotopy fiber over \(a:A\) to be the type

\[
\text{hfiber}(f, a) := \sum_{y:B} \text{Paths}_A(fy, a).
\]
**Definition**

A map \( f : B \rightarrow A \) is a *weak equivalence* if for all \( y : B \), the homotopy fiber \( \text{hfiber}(f, y) \) is contractible.

**Grad Students’ Lemma**

A map \( f : B \rightarrow A \) is a weak equivalence if and only if there exists some map \( g : A \rightarrow B \), inverse to \( f \) in that there are ‘homotopies’

\[
\varepsilon : \prod_{x:A} \text{Paths}_A(fgx, x), \quad \eta : \prod_{y:B} \text{Paths}_B(y, gfy).
\]

**Definition**

Type-theoretic fibrations are projections from dependent sums:

\[
\sum_{x:A} B(x) \rightarrow A
\]
Axiom 2.

∀ satisfies 2-out-of-3 property (i.e. if any two of $f$, $g$, $g \circ f$ are weak equivalences, then so is the third).

This is an easy application of Grad Students’ Lemma.
**Axiom 3.**

Both $\mathcal{F}$ and $\mathcal{W} \cap \mathcal{F}$ are stable under pullback.

Type-theoretically:

$$
\sum_{z:C} B(tz) \rightarrow \sum_{x:A} B(x)
$$

**Lemma**

*For all $a:A$ we have* $\text{hfiber}(\pi_1, a) \simeq B(a)$. 
Axiom 4.

For every object $B \in \mathbf{C}$ there exists a path object i.e. the exists a factorization of the diagonal map $\Delta = (1_B, 1_B) : B \to B \times B$:

$$
\begin{array}{ccc}
\text{Paths}(B) & \xrightarrow{\sigma} & B \\
\downarrow & & \downarrow \Delta \\
B & \xrightarrow{\Delta} & B \times B
\end{array}
$$

where $\sigma$ is a weak equivalence and $p$ is a fibration.

Take:

$$
\text{Paths}(B) = \sum_{x:B} \sum_{y:B} \text{Paths}_B(x, y)
$$

with $\sigma(x) = (x, x, \text{refl}_B(x))$
Some properties and constructions

**Factorization Lemma**

Every map $f : B \rightarrow A$ admits a factorization $f = p \circ \sigma$, where $\sigma \in \mathcal{W}$ and $p \in \mathcal{F}$.

**Right properness**

The pullback of a weak equivalence along a fibration is again a weak equivalence.
Given a diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
B & \xrightarrow{g} & C
\end{array}
\]

its homotopy pullback is defined as:

\[
\sum_{x:A} \sum_{y:B} \text{Paths}_C(fx, gy).
\]

We also constructed some less trivial limits, however we are still working on a general construction of a homotopy limit.
Related work:

- Semantics of type theory in fibration categories (Arndt + K.).
- Model structure (given by HITs) as the basic framework to develop homotopy limits (Lumsdaine).
- Homotopy colimits in type theory + Univalence Axiom (Voevodsky).
15 minutes are probably over by now...