Interpolation error in DNS simulations of turbulence: consequences for particle tracking

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Where innovation starts
Content

- Introduction
- Interpolation
  - Performance
  - Consequences
- Summary and outlook

Introduction

Fluid
- Isotropic turbulence
- DNS simulations
- Pseudo spectral code
- Tri-periodic domain

Particles
- Lagrangian tracking
- One-way coupling
- Small light spherical particles
- Maxey & Riley equations
Introduction

- Light particles
- Bed-load sediments
  - Pattern formation
  - Turbidity currents
- Plankton aggregates
Criteria for interpolation methods

• High order of convergence
• High order of smoothness
• Small number of FFTs
• Small overall errors
## Interpolation methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Order of convergence</th>
<th>Order of smoothness</th>
<th>FFT</th>
<th>Overall errors</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lagrange Interpolation</td>
<td>N-1</td>
<td>0</td>
<td>1</td>
<td>-</td>
<td>For even N</td>
</tr>
<tr>
<td></td>
<td>N-1</td>
<td>-1</td>
<td>1</td>
<td>-</td>
<td>For odd N</td>
</tr>
<tr>
<td>Spline interpolation</td>
<td>N-2</td>
<td>(N-2)/2</td>
<td>1</td>
<td>-</td>
<td>Only even N</td>
</tr>
<tr>
<td>Hermite interpolation</td>
<td>N-1</td>
<td>(N-2)/2</td>
<td>8</td>
<td>+</td>
<td>Only even N</td>
</tr>
<tr>
<td>B-spline interpolation</td>
<td>N-1</td>
<td>N-2</td>
<td>1</td>
<td>+</td>
<td></td>
</tr>
</tbody>
</table>
B-spline functions

\[ B_{(1)} \]

\[ B_{(2)} \]

\[ B_{(3)} \]

\[ B_{(4)} \]
B-spline interpolation

- 2 steps:
  - transformation to B-spline basis
  - carry out the interpolation
- Transformation efficiently done in Fourier space
- Fast algorithm for interpolation
Relative interpolation error

![Graph showing relative interpolation error for different methods: linear, Lagrange, spline, Hermite, and B-spline. The x-axis represents $k\Delta x$ and the y-axis represents error on a log scale.]
Discretisation error

- Re=275
- $K_{\text{max}} \eta = 1.8$
- $K_{\text{max}} \eta = 3.7$
Error interpolation

The graph shows the error interpolation for different types of interpolation methods: linear, Lagrange, spline, Hermite, and B-spline. The x-axis represents the parameter $k$, and the y-axis represents the error on a logarithmic scale. The graph illustrates how each method's error varies with $k$. For example, the linear method shows a gradual increase in error, while the Lagrange method displays a more complex behavior. The spline method exhibits a different pattern, and the Hermite and B-spline methods have distinct error profiles as well.
Comparison errors

- Error ratio = \frac{\text{Interpolation error}}{\text{Discretisation error}}

- Linear inter. (N=2): \quad 3.7
- Lagrange inter. (N=4): \quad 0.50
- Spline inter. (N=4): \quad 0.63
- Hermite inter. (N=4): \quad 0.06
- B-spline inter. (N=4): \quad 0.08
Lagrangian error

- Testing the theory
- Time evolvement of error
- Evolving tracers with different interpolation methods
- Comparison with a simulation of double grid resolution
Lagrangian error of tracers

- Linear N=2
- B-spline N=2
- B-spline N=4
- $L^4$
- $L^2$
- $L^1$

The graph shows the error as a function of time ($t$) on a logarithmic scale for different methods and norms.
Lagrangian error of tracers

![Graph showing error with different models and time]

- Linear $N=2$
- B-spline $N=2$
- B-spline $N=4$
- $L^4$
- $L^2$
- $L^1$
Lagrangian error at Kolmogorov time scale

\[ \log_{10}(error) \]

- Lagrange
- Spline
- B-spline
- Hermite

\( N \) vs. error

- Error decreases with increasing \( N \)
- Lagrange method shows the least error
- Hermite method has a higher error compared to Lagrange
Lagrangian error double grid resolution

![Graph showing error versus N (natural logarithmic scale) for different methods: Lagrange, Spline, B-spline, Hermite. The x-axis represents N, and the y-axis represents error. The graph illustrates the improved performance of Hermite and B-spline methods over Lagrange and Spline methods as N increases.](image-url)
Acceleration of particles

- Important for the calculation of forces on the particles
- Important for statistics
Acceleration of tracers

- linear $N=2$
- Lagrange $N=4$
- spline $N=4$
- B-spline $N=4$
- Hermite $N=4$
Acceleration of tracers

![Graph showing the acceleration of tracers with different interpolation methods and orders. The graph plots acceleration (a) against time (t) for linear, Lagrange, spline, B-spline, and Hermite interpolation methods. The graph includes a zoomed-in inset to highlight the behavior at specific time points.]
Acceleration spectrum

- Linear $N=2$
- Lagrange $N=4$
- Spline $N=4$
- B-spline $N=4$
- Hermite $N=4$
- Spectral
Acceleration spectrum

The graph shows various acceleration spectrum curves for different methods:
- Linear N=2
- Lagrange N=4
- Spline N=4
- B-spline N=4
- Hermite N=4
- Spectral

The x-axis represents the wave number (k), and the y-axis represents the acceleration spectrum (acc spectrum) on a logarithmic scale.
Energy in spectrum double grid resolution

![Graph showing energy in spectrum double grid resolution with different curves for Lagrange, Spline, B-spline, and Hermite methods.]
Optimal interpolation method

\[ k_{\text{max}} = \frac{1}{3} N_g \]
Conclusions

• Advantages B-spline interpolation
  - High order of convergence
  - High order of smoothness
  - Low number of FFTs needed
  - Errors comparable with Hermite interpolation


• Methods for estimation interpolation error
  - Compare interpolation error with discretisation error
  - Linear interpolation not sufficient for our purpose
Questions ?
Optimal interpolation method

- Criteria: Error ratio < 1

\[ k_{\text{max}} = \frac{\sqrt{2}}{3} N_g \]
\[
\text{real}(U_k) \quad \text{F}(U_k)
\]

\[
D \quad \Delta x, 2\Delta x \quad \text{F}(D)
\]

\[
\text{real}(U_kD) \quad \text{F}(U_kD)
\]
\[
\text{real}(U_k D) \quad \text{F}(U_k D)
\]

\[
\begin{align*}
\text{C} & \quad \text{F}(C) \\
\text{real}((U_k D) \ast C) & \quad \text{F}((U_k D) \ast C)
\end{align*}
\]
3D interpolation

- Sequence of 1D interpolations: fast
- 2 properties needed
- Superposition

Maxey & Riley equations

\[ m_p \frac{du_p}{dt} = 6 \pi a \mu \left( u - u_p + \frac{1}{6} a^2 \nabla^2 u \right) + m_f \frac{Du}{Dt} - (m_p - m_f)g \hat{z} \]

\[ + \frac{1}{2} m_f \left( \frac{Du}{Dt} - \frac{du_p}{dt} + \frac{1}{10} a^2 \frac{d}{dt} (\nabla^2 u) \right) \]

\[ + 6 a^2 \rho \sqrt{\pi \nu} \int_{-\infty}^{t} K_B(t - \tau) \frac{df}{dt}(\tau) d\tau \]

\[ = F_{St} + F_P + F_G + F_{AM} + F_B. \]
Maxey & Riley equations

\[
\frac{m_p}{dt} \frac{du_p}{dt} = 6\pi a \mu \left( u - u_p + \frac{1}{6} a^2 \nabla^2 u \right) + m_f \frac{Du}{Dt} - (m_p - m_f) g \hat{z} \\
+ \frac{1}{2} m_f \left( \frac{Du}{Dt} - \frac{du_p}{dt} + \frac{1}{10} a^2 \frac{d}{dt}(\nabla^2 u) \right) \\
+ 6a^2 \rho \sqrt{\pi} \nu \int_{-\infty}^{t} K_B(t - \tau) \frac{df}{d\tau}(\tau) d\tau
\]

\[
= F_{St} + F_P + F_G + F_{AM} + F_B.
\]


\[
K_B(t) = \frac{1}{\sqrt{t}}.
\]

\[
f(t) = u - u_p + \frac{1}{6} a^2 \nabla^2 u,
\]
Interpolation

High order interpolation vs linear interpolation

Accuracy vs Speed

Interpolation error vs discretisation error
Error for B-spline interpolation

- $t_e$ = after one eddy turnover time
- $t_K$ = at Kolmogorov time scale