Methods and Techniques for Multifractal Spectrum Estimation in Financial Time Series

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Abstract. In this paper, we compare two key approaches used in time series analysis, namely the Multifractal Detrended Fluctuation Analysis and Multifractal Diffusion Entropy Analysis. The comparison is done from both the theoretical and numerical point of view. To put some flesh on bare bones, we illustrate our analysis by applying both methods to three model time series. As a fourth illustration we analyze empirical time series of daily returns of S&P500 stock index recorded over the 50 years period. We argue that while the Multifractal Detrended Analysis is computationally more efficient, the Multifractal Diffusion Entropy Analysis is conceptually cleaner. In addition, the latter allows a wider applicability in cases when time series have underlying distributions that are heavy tailed.

Keywords: Multifractal spectrum, Detrended Fluctuation Analysis, Rényi Entropy.

1 Introduction

Scaling properties belong among the most important quantifiers of complexity in many real systems, including chaotic dynamical systems, biological systems and financial markets. The presence of scaling usually points to a non-trivial cooperative behavior imprinted in temporal correlations. Techniques of fractal geometry can be then applied to reveal and analyze the potential scaling exponents. In practice, however, it is rather common that systems exhibit a multiple scaling. In such a case the methods of multifractal analysis can be conveniently employed. The concept of scaling has been used routinely in theory of critical phenomena (e.g., phase transitions) and in chaotical dynamical systems since a long time. Applications in stochastic processes and ensuing time series (including financial time series) are, however, of newer vintage. At present, there are a number of works addressing this line of research. These range from seminal papers of Hurst [1], through theory of fractal geometry [2] and fractional dynamics [3] to the multifractal calculus [4–6]. Particularly, in the theory of multifractals the key rôle is played by the notion of Rényi’s entropy and ensuing generalized dimensions [7]. Recently, also the concept of Rényi’s transfer entropy [8] has been invoked in this context.

Our focus in this paper will be the study of time series from the multifractal point of view. As a rule, the presence of multifractality signalizes that the time series exhibits a complex behavior with long-range time correlations manifested on different intrinsic time scales. When considering real financial data series such as empirical series from stock market, the multifractality points, e.g., to the onset of crises, economical cycles and other non-linear phenomena that
cannot be modeled with Wiener process [9]. In this paper we compare two key techniques, namely, Multifractal Detrended Fluctuation Analysis (MF-DFA) and Multifractal Diffusion Entropy Analysis (MF-DEA). In particular MF-DEA is discussed in terms Rényi’s entropy. Our analysis reveals that while the MF-DFA is computationally efficient, the MF-DEA is conceptually cleaner. We also emphasize the fact that the MF-DEA is better suited to discuss cases when time series have underlying heavy tailed distributions.

The paper is organized as follows. In Section 2 we briefly outline some fundamentals of the multifractal calculus. In Section 3 we define a MF-DFA method. The related salient theoretical issues are presented in Section 4. Section 5 is dedicated to the MF-DEA method. Comparison of both methods for monofractal series is done in Section 6. Finally, in Section 7 are all methods demonstrated on four selected time series.

2 Some fundamentals of multifractal analysis

One way to reveal a multiscaling structure in time series is to construct a multifractal spectrum, which catalogues the involved scaling exponents. In the following we will use the standard definition, presented, e.g., in Ref. [10]. To this end we consider a time series \( \{x_i\}_{i=1}^{N} \) where \( i \) denotes a discrete times of the evolution with some specific time lag \( \tau \). We examine the scaling of the probability in different regions. The probability of a region in the space is given as \( p_i = \lim_{N \to \infty} \frac{N_i}{N} \), where \( N_i \) is number of points lying in the region and \( N \) is total number of points. We assume that probabilities scale with some scaling exponent (Lipschitz–Hölder exponent) \( \alpha \) as \( p_i \propto l^{\alpha} \). For different regions we can obtain generally different values of \( \alpha \). Our aim is to describe a distribution of \( \alpha \), which is assumed to be in a form \( d\alpha \rho(\alpha) \), where \( f(\alpha) \) represents a fractal dimension of the subset scaling with \( \alpha \). Moreover, it can be introduced a total sum of all \( p_i^q \) — partition function, which scales with the scaling exponent \( \tau(q) \), so that

\[
Z(q, l) = \sum_i p_i^q \sim l^{\tau(q)}. \tag{1}
\]

The relation between \( f(\alpha) \) and \( \tau(q) \) is given by substitution of probability distribution of scaling exponents into the \( Z(q, l) \):

\[
Z(q, l) = \int d\alpha \rho(\alpha) l^{-f(\alpha)} l^{q\alpha} = \int d\alpha \rho(\alpha) l^{q\alpha} - f(\alpha) \sim l^{\tau(q)}. \tag{2}
\]

By using the method of stationary phase one arrives at the equation

\[
\tau(q) = q\alpha(q) - f(\alpha(q)), \tag{3}
\]

where \( \alpha(q) \) is value for which the expression \( q\alpha - f(\alpha) \) is extremal. Under assumption that \( f(\alpha) \) is differentiable the latter is equivalent to the condition that \( f'(\alpha) = q \). Eq. (3) then implies that \( \tau(q) = \alpha \) and so \( \tau(q) \) is nothing but the Legendre transform of \( f(\alpha) \). The generalized dimension is defined as

\[
D_q = \lim_{l\to0} \frac{1}{q-1} \frac{\ln Z(q, l)}{\ln l} = \frac{\tau(q)}{q-1}, \tag{4}
\]
and is a generalization of a support dimension \( q = 0 \), fractal dimension \( q = 1 \) and correlation dimension \( q = 2 \), see Ref. [7]. The term \( S_q = \frac{1}{q-1} \ln Z(q, l) \) is known as the Rényi entropy and it represents a one-parametric generalization of Shannon’s entropy (to which it reduces for \( q \to 1 \)). For different \( q \)'s we accentuate in the sum (1) different values of involved probabilities and so we can zoom to different regions of the distribution.

Note that if we rewrite the partition function as

\[
Z(q, l) = \sum_i p_i q_i^{q-1} = \langle P^{q-1} \rangle ,
\]

the generalized dimension simply equals the scaling exponent of the generalized average of the probability distribution, namely \( s \cdot \sqrt{(P^{q-1})} \sim l^{D_q} \).

An alternative way to quantify a multiscaling behavior in time series is to generalize the concept of the Hurst exponent. The Hurst exponent is defined as the scaling exponent of the average increments of length of a dynamical process, i.e. \( \langle |x(t + \tau) - x(t)| \rangle \sim \tau^H \). When investigating the \( q \)-th moment of these increments, one may define the scaling:

\[
\langle |x(t + \tau) - x(t)|^q \rangle \sim \tau^{\zeta(q)} .
\]

For monofractal processes with the Hurst exponent \( H \), the \( \zeta(q) \) scaling exponent typically equals to \( \zeta(q) = H q \). For multifractals \( \zeta(q) \) is not simply linear in \( q \) which motivates the definition of the generalized Hurst exponent \( H(q) \) as \( H(q) = \zeta(q)/q \). In this case the following scaling relation holds

\[
\sqrt{\langle |x(t + \tau) - x(t)|^q \rangle} \sim \tau^{H(q)}.
\]

For \( q = 1 \) this reduces to the standard Hurst scaling with \( H(q = 1) = H \).

### 3 Multifractal Detrended Fluctuation Analysis

The classical way how to calculate the multifractal spectrum is MF-DFA [11, 12]. The algorithm is following: if we have a noise-like series, we transform it into the walk-like series by a cumulative sum \( Y(n) = \sum_{i=1}^n (x_i - \langle x \rangle) \). The subtraction of the mean value is not important for the method itself, but it is crucial for the relation to the classical multifractal analysis, as discussed further. The method begins with dividing the series into \( N_s \) parts with length \( s \). Local linear (quadratic, cubic,...) trends \( y_\nu \) are estimated. The cornerstone of this method is fluctuation function which represents a total deviation from the trend and for each element is calculated as

\[
F(\nu, s)^2 = \sum_{i=1}^s (Y(s(\nu - 1) + i) - y_\nu(i))^2 .
\]

MF-DFA method, which is a generalization of the monofractal DFA [12], deals with generalized fluctuation function dependent on \( q \) and defined as

\[
F(q, s) = \left\{ \frac{1}{N_s} \sum_{\nu=1}^{N_s} [F(s, \nu)^2]^{q/2} \right\}^{1/q} ,
\]

for \( q = 1 \) this reduces to the standard Hurst scaling with \( H(q = 1) = H \).
which is nothing else than $\sqrt{\langle F(s,\cdot)^q \rangle}$, where we average over all time intervals. The fluctuation function satisfies the scaling property $F(q,s) \propto s^{h(q)}$. Because $N_s = N/s$, the following sum scales as

$$\left\{ \sum_{\nu=1}^{N_s} [F(\nu,s)^2]^{q/2} \right\} \sim s^{q h(q) - 1}.$$  \hspace{1cm} (10)

In passing, we may note that the exponent $h(2)$ is the exponent obtained from standard monofractal Detrended Fluctuation Analysis, while the exponent $h(1)$ is related to the R/S analysis and in the case of monofractal scaling both should be equal to the Hurst exponent $H$, cf. Ref. [12].

If we consider only a stationary, normalized and positive series $x_i$, it is possible to omit detrending procedure and we obtain that

$$\sum_{\nu=1}^{N_s} F^2_{FA}(\nu,s) = \sum_{\nu=1}^{N_s} |Y(\nu s) - Y((\nu - 1)s)|^q \sim s^{q h(q) - 1}.$$  \hspace{1cm} (11)

In order to relate the method to multifractal analysis, we define a partition sum

$$Z_q(s) = \sum_{\nu=1}^{N_s} |p_s(\nu)| \sim s^{\tau_M(q)},$$  \hspace{1cm} (12)

where $\tau_M(q)$ is the scaling function related to multifractal spectrum via Legendre transform and $p_s(\nu) = \sum_{k=(\nu-1)s+1}^{\nu s} x_k = (Y(\nu s) - Y((\nu - 1)s))$, see Ref. [12]. Eventually, we get the relation between $\tau_M$ and $h$

$$\tau_M(q) = q h(q) - 1.$$  \hspace{1cm} (13)

4 Problems of $\tau_M(q)$

We have seen that the relation for $\tau_M(q)$ is problematic as it cannot be compared to standard multifractal analysis for every series. Let us now assume that we have a stationary, normalized and positive series $x_k$. Thus, the $p_s(\nu)$ should have the form $p_s(\nu) = \sum_{k=(\nu-1)s+1}^{\nu s} x_k = (Y(\nu s) - Y((\nu - 1)s))$, [13]. So, $(x - \langle x \rangle)$ represents a measure that generates the probability density $|p_s|$. The problem is that this measure is not a proper measure, because it does not obey axioms of measure, because we can find such set that for $A \subseteq B$ is $\mu(A) > \mu(B)$ [13]. Therefore, it can happen that relations derived in the last section might be not completely right. Apart from that, $D_q = q h(q) - 1$ which for monofractal series, where $h(q) = H$, is equal to $D_q = q H - 1$. This leads to a singularity in $q = 1$, unless $H = 1$.

One possible way how to overcome these problems is consider the scaling function in the following form [13]:

$$\tau(q) = D(q - 1) - K(q),$$  \hspace{1cm} (14)

where the function $K(q)$ is a cumulant generating function and is related to so called fractal co-dimension and equals to $H'q - \zeta(q)$, where $H'$ is a constant to
be determined [14]. The relation between $\zeta(q)$ and $h(q)$ from MF-DFA analysis is $\zeta(q) = qH(q) = q(h(q) - 1)$. Finally, from the relation $\tau(1) = 0$, it is possible to determine the $H'$, which is equal to $H' = h(1) - 1$. For time series, we know that $D_0 = -\tau(0)$ is equal to dimension of support, which is equal to one [13], so the formula (14) takes the form

$$\tau_G(q) = qh(q) - qH' - 1 = q(h(q) - h(1) + 1) - 1.$$  \hspace{2cm} (15)

5 Diffusion Entropy Analysis and MF-DEA

The problem of MF-DFA is that in case of multifractality that originates from power-law scaling of distributions is the application of the method inappropriate, simply because the variance (and higher moments) of the time series is infinite. Of course, the real time series are finite, and hence the empirical variance is finite, but because of its non-trivial dependence of time-lag time, it does not give right predictions. In such cases it is more suitable to analyze multifractality via the self-similarity scaling ansatz. If we assume that the probability distribution of a time series has the form (as is the case, e.g., for Gaussian noise or Lévy processes)

$$p(x,t) = \frac{1}{t^\delta} F\left(\frac{x}{t^\delta}\right),$$  \hspace{2cm} (16)

then it is possible to estimate $\delta$ via Shannon’s entropy [15]

$$S(t) = -\int dx p(x,t) \ln[p(x,t)],$$  \hspace{2cm} (17)

because in this case $S(t) = A + \delta \ln t$ ($A$ is a constant).

The monofractal Diffusion Entropy Algorithm is based on the fluctuation collection. Let us have a noise-like series $\xi_i$ and let us define $x_\tau(t) = \sum_{i=1}^{\tau} \xi_{i+t}$. Then we divide all values of $x_\tau$ into boxes of length $\epsilon$ and calculate the probability of each box as $p_i(\tau) = \frac{N_i(\tau)}{N_{\tau+1}}$ where $N_i(\tau)$ is number of $x_\tau(t)$ that have values in the $i$-th box. The ensuing Shannon’s entropy is of the form

$$S(\tau) = -\sum_i p_i(\tau) \ln[p_i(\tau)].$$  \hspace{2cm} (18)

The MF-DEA is a direct generalization of the previous monofractal version. If we instead of Shannon entropy use the whole class of Rényi entropies, we get a class of scaling exponents [16]

$$S_q(\tau) = B_q + H(q) \ln \tau.$$  \hspace{2cm} (19)

Here $B_q$ is $\tau$-independent constant and $H(q)$ is the generalized Hurst exponent.

6 Comparison of $\tau(q)$’s for monofractal series

So far, we have been dealing with two different scaling functions $\tau(q)$. The first one is provided via Mandelbrot’s scaling definition [2] as $\langle |X(t)|^q \rangle \propto t^{\tau_M(q)+1}$. 


The second one results from the scaling of the empirical distribution and the best way to phrase it is terms of the generalized dimension as $\tau(q) = D_q(q - 1)$ and ensuing Rényi entropy as $\tau(q) \simeq -\ln \langle P^q \rangle / \ln t$ and therefore $\langle P^q \rangle \propto t^{-\tau_q(q)}$. Let us consider a self-similar monofractal process with probability density in the form $p(x,t)dx = [F(x/t^H)/t^H]dx$ and calculate scaling functions in both cases

$$\langle |X(t)|^q \rangle = \int dx |x|^q p(x,t) = \int dx |x|^q \frac{1}{t^H} F\left(\frac{x}{t^H}\right) = c(q) t^{qH}.$$  

(20)

Since $c(q)$ is $t$-independent, we have $\tau_M(q) = qH - 1$. The observant reader will recall that we have noticed this behavior already in Section 3. In the case of Rényi entropy we have

$$\langle |P(t)|^{q-1} \rangle = \int dx p(x,t)^q = \int dx \frac{1}{t^H} \left[ F\left(\frac{x}{t^H}\right) \right]^q = t^{H(1-q)} \int dy \ [F(y)]^q = c'(q) t^{H(1-q)}. $$  

(21)

The $t$-independence $c'(q)$ then implies that $\tau_R(q) = H(q - 1)$. By comparing the scalings (20) and (21) we obtain the following relation

$$\tau_M(q) = \frac{q}{q - 1} \tau_R(q) - 1.$$  

(22)

7 Numerical comparison of multifractal techniques

To put some flesh on the bare bones we shall now compare the above methods on explicit examples. To this end we chose four sample series, three of an artificial kind and one corresponding to an empirical financial series. The first series corresponds to a Brownian motion, the second to a monofractal series of a Fractional Brownian motion (FBM) with the Hurst exponent $H = 0.75$, the third is an artificial multifractal process (MFP), generated from a binomial cascade (see, e.g., Ref. [17]). Finally, the last series represents the daily records of the Standard and Poor’s 500 (S&P500) financial index gathered over the period of 50 years. Log returns of these time series are plotted on Fig. 1.

On Fig. 2 we depict results of all techniques used. In case of Detrended Analysis Method (top left figure) we can observe that the spectrum of S&P500 index is similar to the Gaussian spectrum and the spectrum of MFP, which points to the fact that most of the time is the index in a “Gaussian” regime, and only in the times of crises, etc., the large fluctuations can be observed and therefore is the spectrum wider (especially in the left part). The fractional Brownian motion was purposely chosen so, that all scaling exponents are shifted to the larger values. We shall note that in the theoretical case we should observe in the case of Brownian motion and FBM a singular spectrum with a single point that corresponds to its Hurst exponent. Nevertheless, in practice we do not obtain a singular spectrum. On the top right figure is displayed a spectrum obtained from a generalized scaling function. In this case,
the spectrum serves to disclose deviations from monofractal behavior and we observe that the lines of Brownian motion and FBM practically coincide, while spectrum of S&P500 and foremost MFP reveal rich underlying multifractal behavior. The bottom left figure shows the scaling exponents of Rényi’s entropy, where we can again observe the shift of FBM, the multifractality of the MFP and the fact that S&P500 reveal a multifractal behavior, but not as strong as the artificial series. Finally, the bottom right picture shows the estimated generalized Hurst exponent. Unfortunately, for negative values of $q$ the method is unreliable due to well known instabilities in Rényi’s entropy [7]. For positive values we get a steady behavior for all discussed series, but in the case of S&P500, we see a decreasing function even for positive values. The latter signalizes that though that the series does not have a wide spectrum as, e.g., MFP, the spectrum is rounder than for other ones and scaling exponents play an important rôle.

All in all, by comparing all above methods, the most reliable method (at least from the theoretical point of view) is the Multifractal Entropy Analysis. This is because of its clear interpretation of the scaling exponent, and its stability and validity also for processes with infinite higher moments. The disadvantage of the method is the computational robustness of the fluctuation collection that is of order $O(n^2)$. On the other hand, Multifractal Detrended Fluctuation Analysis is computationally very efficient and apart from theoretical issues related to the interpretation of the scaling exponents, it provides a clear numerical tool allowing to classify the multifractality.
8 Conclusions

In this paper we have compared two key methods used in time series analysis; the Multifractal Detrended Fluctuation Analysis and Multifractal Diffusion Entropy Analysis. We have demonstrated that MF-DEA based on Rényi entropy represents a pertinent tool that allows to conveniently quantify and qualify complex structures present in numerous realistic time series, and more specifically in financial time series. We have shown under what conditions both outlined methods can successfully estimate multifractal exponents, multifractal spectrum as well as scaling functions, and when the corresponding results can be naturally translated to each other. Added advantage of the MF-DEA stems from the fact that one may rigorously discuss also systems with infinite higher moments.

By using the example of S&P500 index, we have seen that corresponding empirical time series posses interactions that are highly nonlinear, and long-ranged. This is a clear manifestation of a number of interlocked driving dynamics operating at different time scales each with its own scaling function. In an econophysic such a behavior typically points to the presence of recurrent
economic cycles, crises, large fluctuations (i.e., marginal events such as spikes or sudden jumps), and other non-linear phenomena that are out of reach of more conventional multivariate methods [9].

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References