Two-sided Rauzy Induction

Francesco Dolce

Leiden, 23\textsuperscript{th} January 2014

Joint work with

V. Berthé\textsuperscript{1}, C. De Felice\textsuperscript{2}, D. Perrin\textsuperscript{3}, C. Reutenauer\textsuperscript{4} and G. Rindone\textsuperscript{3}

\textsuperscript{1}Université Paris Diderot, \textsuperscript{2}Università di Salerno, \textsuperscript{3}Université Paris Est, \textsuperscript{4}Université du Quebec à Montreal
Outline

1. Interval exchange transformations
   - Interval exchange transformations
   - Regular interval exchange transformations

2. Rauzy induction
   - Right Rauzy induction
   - Two-sided Rauzy induction

3. Natural coding
   - Natural coding
   - Regular interval exchange sets
   - Return theorem
Outline

1. Interval exchange transformations
   - Interval exchange transformations
   - Regular interval exchange transformations

2. Rauzy induction
   - Right Rauzy induction
   - Two-sided Rauzy induction

3. Natural coding
   - Natural coding
   - Regular interval exchange sets
   - Return theorem
Interval exchange transformations

Let \((A, <)\) be an ordered set and let \((I_a)_{a \in A}\) be an ordered partition of \([\ell, r]\).
A *interval exchange transformation* is a function \(T : [\ell, r] \rightarrow [\ell, r]\) defined by

\[
T(z) = z + \alpha_z \quad \text{if } z \in I_a.
\]
Regular interval exchange transformations

$T$ is said to be \textit{minimal} if for any $z \in [\ell, r[$ the orbit $\mathcal{O}(z) = \{ T^n(z) \mid n \in \mathbb{Z} \}$ is dense in $[\ell, r[$.
Regular interval exchange transformations

$T$ is said to be \textit{minimal} if for any $z \in [\ell, r[$ the orbit $\mathcal{O}(z) = \{ T^n(z) \mid n \in \mathbb{Z} \}$ is dense in $[\ell, r[$.

$T$ is said \textit{regular} if the orbits of the separation points $\neq \ell$ are infinite and disjoint.

**Theorem [Keane, 1975]**

A regular interval exchange transformation is minimal.
A regular interval exchange transformation is minimal.

\( T \) is said to be \textit{minimal} if for any \( z \in [\ell, r[ \) the orbit \( \mathcal{O}(z) = \{ T^n(z) \mid n \in \mathbb{Z} \} \) is dense in \( [\ell, r[ \).

\( T \) is said \textit{regular} if the orbits of the separation points \( \neq \ell \) are infinite and disjoint.

\textbf{Theorem [Keane, 1975]}

A regular interval exchange transformation is minimal.
Proposition

Let $T$ be a regular $s$-interval exchange transformation. Then $T^n$ is a regular $n(s - 1) + 1$-interval exchange transformation.
Proposition

Let $T$ be a regular $s$-interval exchange transformation. Then $T^n$ is a regular $n(s - 1) + 1$-interval exchange transformation.
Proposition

Let $T$ be a regular $s$-interval exchange transformation. Then $T^n$ is a regular $n(s - 1) + 1$-interval exchange transformation.
Proposition

Let $T$ be a regular $s$-interval exchange transformation. Then $T^n$ is a regular $n(s - 1) + 1$-interval exchange transformation.

$$J_b = T(I_b) \quad \alpha \quad J_a = T(I_a)$$

$$J_{ab} \quad J_{ba} \quad J_{aa}$$

$$J_{a_0a_1\ldots a_{m-1}} = T^m(I_{a_0}) \cap T^{m-1}(I_{a_1}) \cap \ldots \cap T(I_{b_{a-1}}) \quad \text{and} \quad I_w = T^{-|w|}(J_w)$$
Outline

1. Interval exchange transformations
   - Interval exchange transformations
   - Regular interval exchange transformations

2. Rauzy induction
   - Right Rauzy induction
   - Two-sided Rauzy induction

3. Natural coding
   - Natural coding
   - Regular interval exchange sets
   - Return theorem
Admissible semi-intervals

Let $T$ be an interval exchange transformation on the semi-interval $[\ell, r[$. For $\ell < t < r$, the semi-interval $[\ell, t]$ is right-admissible for $T$ if there is a $k \in \mathbb{Z}$ s.t. $t = T^k(\gamma_a)$ for some $a \in A$ and:

(i) if $k > 0$, then $t < T^h(\gamma_a)$ for all $0 < h < k$,

(ii) if $k \leq 0$, then $t < T^h(\gamma_a)$ for all $k < h \leq 0$, 
Let $T$ be an interval exchange transformation on the semi-interval $[\ell, r[$.
For $\ell < t < r$, the semi-interval $[\ell, t[$ is right-admissible for $T$ if there is a $k \in \mathbb{Z}$ s.t. $t = T^k(\gamma_a)$ for some $a \in A$ and:

(i) if $k > 0$, then $t < T^h(\gamma_a)$ for all $0 < h < k$,
(ii) if $k \leq 0$, then $t < T^h(\gamma_a)$ for all $k < h \leq 0$,
Admissible semi-intervals

Let $T$ be an interval exchange transformation on the semi-interval $[\ell, r]$.

For $\ell < t < r$, the semi-interval $[\ell, t]$ is right-admissible for $T$ if there is a $k \in \mathbb{Z}$ s.t. $t = T^k(\gamma_a)$ for some $a \in A$ and:

(i) if $k > 0$, then $t < T^h(\gamma_a)$ for all $0 < h < k$,

(ii) if $k \leq 0$, then $t < T^h(\gamma_a)$ for all $k < h \leq 0$, 

\[ \begin{align*}
0 & \quad a & \quad 1-2\alpha & \quad b & \quad 1-\alpha & \quad c & \quad 1 \\
\gamma_c & \quad \alpha & \quad b & \quad c & \quad 2\alpha & \quad a & \quad 1 \\
\end{align*} \]
Let $T$ be an interval exchange transformation on the semi-interval $[\ell, r[.$
For $\ell < t < r$, the semi-interval $[\ell, t[$ is right-admissible for $T$ if there is a $k \in \mathbb{Z}$ s.t. $t = T^k(\gamma_a)$ for some $a \in A$ and:

(i) if $k > 0$, then $t < T^h(\gamma_a)$ for all $0 < h < k$,
(ii) if $k \leq 0$, then $t < T^h(\gamma_a)$ for all $k < h \leq 0$, 

\begin{tikzpicture}

\draw[red] (0,0) -- (1,0) node[below] {$a$} -- (2,0) node[below] {$1 - 2\alpha$} -- (3,0) node[below] {$b$} -- (4,0) node[below] {$1 - \alpha$} -- (5,0) node[below] {$c$} -- (6,0) node[below] {$1$} node[above] {$T(\gamma_c)$};
\draw[blue] (0,-1) -- (1,-1) node[below] {$\alpha$} -- (2,-1) node[below] {$\gamma_c$} -- (3,-1) node[below] {$c$} -- (4,-1) node[below] {$2\alpha$} -- (5,-1) node[below] {$a$} -- (6,-1) node[below] {$\bullet$};
\draw[green] (1,-1) -- (2,-1) node[below] {$b$};
\draw[blue,->] (0.5,-1.5) arc (180:0:0.5) node[left] {$T$};
\end{tikzpicture}
Let $T$ be an interval exchange transformation on the semi-interval $[\ell, r]$.

For $\ell < t < r$, the semi-interval $[\ell, t]$ is right-admissible for $T$ if there is a $k \in \mathbb{Z}$ s.t. $t = T^k(\gamma_a)$ for some $a \in A$ and:

(i) if $k > 0$, then $t < T^h(\gamma_a)$ for all $0 < h < k$,
(ii) if $k \leq 0$, then $t < T^h(\gamma_a)$ for all $k < h \leq 0$,
Let \( T \) be an interval exchange transformation on the semi-interval \([\ell, r]\).
For \( \ell < t < r \), the semi-interval \([\ell, t]\) is **right-admissible** for \( T \) if there is a \( k \in \mathbb{Z} \) s.t. \( t = T^k(\gamma_a) \) for some \( a \in A \) and:

(i) if \( k > 0 \), then \( t < T^h(\gamma_a) \) for all \( 0 < h < k \),

(ii) if \( k \leq 0 \), then \( t < T^h(\gamma_a) \) for all \( k < h \leq 0 \),

![Diagram of interval exchange transformation](image-url)
Induced transformations

Let $T$ be a minimal interval exchange transformation and $I \subset [l, r]$. The transformation induced by $T$ on $I$ is the transformation $S : I \rightarrow I$ defined by

$$S(z) = T^n(z) \quad \text{with} \quad n = \min\{k > 0 \mid T^k(z) \in I\}$$

The semi-interval $I$ is called the domain of $S$, denoted $D(S)$. 
Induced transformations

Let $T$ be a minimal interval exchange transformation and $I \subset [l, r]$. The \textit{transformation induced} by $T$ on $I$ is the transformation $S : I \rightarrow I$ defined by

$$S(z) = T^n(z) \quad \text{with} \quad n = \min\{k > 0 \mid T^k(z) \in I\}$$

The semi-interval $I$ is called the \textit{domain} of $S$, denoted $D(S)$.

\textbf{Theorem [Rauzy, 1979]}

Let $T$ be a regular interval exchange transformation and $I$ a right-admissable interval for $T$. The induced transformation is a regular interval exchange transformation.
Induced transformations

\[ T = [0, 2\alpha] \]

\[ S(z) = \begin{cases} 
T^2(z) & \text{if } 0 \leq z < 1 - 2\alpha \\
T(z) & \text{otherwise}
\end{cases} \]
Right Rauzy induction

Let $T$ be a regular interval exchange transformation on $[\ell, r]$. Set

$$Z(T) = [\ell, \max_a \{\gamma_a, T(\gamma_a)\}].$$

We denote by $\psi(T)$ the transformation induced by $T$ on $Z(T)$. 
Right Rauzy induction

Let $T$ be a regular interval exchange transformation on $[\ell, r]$. Set 

$$Z(T) = [\ell, \max_a \{\gamma_a, T(\gamma_a)\}]$$

We denote by $\psi(T)$ the transformation induced by $T$ on $Z(T)$.

**Theorem [Rauzy, 1979]**

Let $T$ be a regular interval exchange transformation. A semi-interval $I$ is right-admissible for $T \iff I = Z(\psi^n(T))$ for some $n > 0$. In this case, the transformation induced by $T$ on $I$ is $\psi^{n+1}(T)$.

The map $T \to \psi(T)$ is called the *right Rauzy induction*. 
Right Rauzy induction

\[
\begin{align*}
T(\gamma_a) & : 0 \rightarrow 1 - 2\alpha \rightarrow 1 - \alpha \rightarrow 1 \\
\psi(T) & : 0 \rightarrow 1 - 2\alpha \rightarrow 1 - \alpha \rightarrow 2\alpha \\
\psi^2(T) & : 0 \rightarrow 2 - 5\alpha \rightarrow 1 - 2\alpha \rightarrow 1 - \alpha
\end{align*}
\]
The symmetrical notions of *left admissible semi-interval* and *left Rauzy induction*, denoted $\varphi$, are defined similarly.

![Diagram of Rauzy Induction](image)
Two-sided Rauzy induction

Let $T$ be a regular interval exchange transformation. For $\ell \leq u < v \leq r$ we say that the semi-interval $I = [u, v[$ is admissible for $T$ if $u, v \in \text{Div}(I, T) \cup r$ with

$$\text{Div}(I, T) = \bigcup_a \left\{ T^k(\gamma_a) \mid -\rho^-(\gamma_a) \leq k < \rho^+(\gamma_a) \right\}$$

$$\rho^-(z) = \min \left\{ n > 0 \mid T^n(z) \in ]u, v[ \right\}, \quad \rho^+(z) = \min \left\{ n \geq 0 \mid T^{-n}(z) \in ]u, v[ \right\}.$$
Two-sided Rauzy induction

Let $T$ be a regular interval exchange transformation. For $\ell \leq u < v \leq r$ we say that the semi-interval $I = [u, v]$ is admissible for $T$ if $u, v \in \text{Div}(I, T) \cup r$ with

$$\text{Div}(I, T) = \bigcup_{a} \left\{ T^k(\gamma_a) \mid -\rho^{-}(\gamma_a) \leq k < \rho^{+}(\gamma_a) \right\}$$

$$\rho^{-}(z) = \min \left\{ n > 0 \mid T^n(z) \in ]u, v[ \right\}, \quad \rho^{+}(z) = \min \left\{ n \geq 0 \mid T^{-n}(z) \in ]u, v[ \right\}.$$

Theorem [BDDPRR (i.e. us), 2013]

The transformation induced by $T$ on $I$ is a regular interval exchange transformation.

Theorem [BDDPRR (i.e. us), 2013]

$I$ is admissible for $T$ if $I$ is the domain of a $\chi \in \{\varphi, \psi\}^*$. In this case, the transformation induced by $T$ on $I$ is $\chi(T)$.
Two-sided Rauzy induction

\[ T(\gamma_c) \quad c \quad T(\gamma_a) \]

\[ \varphi \circ \psi(T) \]

\[ \varphi^2 \circ \psi(T) \]

Francesco Dolce (Université Paris Est)  Two-sided Rauzy Induction  Leiden, 23\textsuperscript{th} January 2014
Outline

1. Interval exchange transformations
   - Interval exchange transformations
   - Regular interval exchange transformations

2. Rauzy induction
   - Right Rauzy induction
   - Two-sided Rauzy induction

3. Natural coding
   - Natural coding
   - Regular interval exchange sets
   - Return theorem
Natural Coding

Let $T$ be an interval exchange transformation relative to $(l_a)_{a \in A}$. The *natural coding* of $T$ relative to $z \in [\ell, r]$ is the infinite word $\Sigma_T(z) = a_0a_1 \cdots \in A^\omega$ defined by

$$a_n = a \quad \text{si} \quad T^n(z) \in l_a.$$
Natural Coding

Let $T$ be an interval exchange transformation relative to $(I_a)_{a \in A}$. The *natural coding* of $T$ relative to $z \in [\ell, r]$ is the infinite word $\Sigma_T(z) = a_0a_1 \cdots \in A^\omega$ defined by

$$a_n = a \text{ si } T^n(z) \in I_a.$$

**Example**

The *Fibonacci word* is the natural coding of the rotation of angle $\alpha = (3 - \sqrt{5})/2$ relative to the point $\alpha$, i.e. $T(z) = z + \alpha \mod 1$. 

![Diagram of the Fibonacci word and interval exchange transformation](image_url)
Natural Coding

Let $T$ be an interval exchange transformation relative to $(I_a)_{a \in A}$. The natural coding of $T$ relative to $z \in [\ell, r]$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in A^\omega$ defined by

$$a_n = a \quad \text{si} \quad T^n(z) \in I_a.$$

Example

The Fibonacci word is the natural coding of the rotation of angle $\alpha = (3 - \sqrt{5})/2$ relative to the point $\alpha$, i.e. $T(z) = z + \alpha \mod 1$. 

$\Sigma_T(z) = a$
**Natural Coding**

Let $T$ be an interval exchange transformation relative to $(l_a)_{a \in A}$. The *natural coding* of $T$ relative to $z \in [\ell, r]$ is the infinite word $\Sigma_T(z) = a_0a_1 \cdots \in A^\omega$ defined by

$$a_n = a \text{ si } T^n(z) \in l_a.$$ 

**Example**

The *Fibonacci word* is the natural coding of the rotation of angle $\alpha = (3 - \sqrt{5})/2$ relative to the point $\alpha$, i.e. $T(z) = z + \alpha \mod 1$. 

$$\Sigma_T(z) = ab$$
Natural Coding

Let $T$ be an interval exchange transformation relative to $(l_a)_{a \in A}$. The *natural coding* of $T$ relative to $z \in [\ell, r]$ is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots \in A^\omega$ defined by

$$a_n = a \quad \text{si} \quad T^n(z) \in l_a.$$

Example

The *Fibonacci word* is the natural coding of the rotation of angle $\alpha = (3 - \sqrt{5})/2$ relative to the point $\alpha$, i.e. $T(z) = z + \alpha \mod 1$. 

$$\Sigma_T(z) = a b a$$
Natural Coding

Let $T$ be an interval exchange transformation relative to $(l_a)_{a \in A}$. The *natural coding* of $T$ relative to $z \in [\ell, r]$ is the infinite word $\Sigma_T(z) = a_0a_1 \cdots \in A^\omega$ defined by

$$a_n = a \quad \text{if} \quad T^n(z) \in l_a.$$

**Example**

The *Fibonacci word* is the natural coding of the rotation of angle $\alpha = (3 - \sqrt{5})/2$ relative to the point $\alpha$, i.e. $T(z) = z + \alpha \mod 1$. 

$$\Sigma_T(z) = a b a a$$
Natural Coding

Let $T$ be an interval exchange transformation relative to $(l_a)_{a \in A}$. The natural coding of $T$ relative to $z \in [\ell, r]$ is the infinite word $\Sigma_T(z) = a_0a_1\cdots \in A^\omega$ defined by

$$a_n = a \; \text{ si } \; T^n(z) \in l_a.$$ 

Example

The Fibonacci word is the natural coding of the rotation of angle $\alpha = (3 - \sqrt{5})/2$ relative to the point $\alpha$, i.e. $T(z) = z + \alpha \mod 1$. 

\[
\Sigma_T(z) = a \; b \; a \; a \; b
\]
Natural Coding

Let $T$ be an interval exchange transformation relative to $(I_a)_{a \in A}$. The *natural coding* of $T$ relative to $z \in [\ell, r]$ is the infinite word $\Sigma_T(z) = a_0a_1\cdots \in A^\omega$ defined by

$$a_n = a \quad \text{si} \quad T^n(z) \in I_a.$$ 

**Example**

The *Fibonacci word* is the natural coding of the rotation of angle $\alpha = (3 - \sqrt{5})/2$ relative to the point $\alpha$, i.e. $T(z) = z + \alpha \mod 1$. 

$$\Sigma_T(z) = a\ b\ a\ a\ b\ a\ \cdots$$

Francesco Dolce (Université Paris Est) Two-sided Rauzy Induction Leiden, 23$^{th}$ January 2014 17 / 23
**Proposition**

If $T$ is minimal, $\mathcal{L}(\Sigma_T(z))$ does not depend on $z$.

When $T$ is regular (minimal), $\mathcal{L}(T) = \mathcal{L}(\Sigma_T(z))$ is said a *regular (minimal) interval exchange set* (*linear complexity, neutrality, tree set* \(^1\), *finite index basis, etc.*).

---

1. See Valérie Berthé’s talk next week.

---

Francesco Dolce (Université Paris Est)  
Two-sided Rauzy Induction  
Leiden, 23\textsuperscript{th} January 2014
Regular interval exchange sets

Proposition
If $T$ is minimal, $\mathcal{L}(\Sigma_T(z))$ does not depend on $z$.

When $T$ is regular (minimal), $\mathcal{L}(T) = \mathcal{L}(\Sigma_T(z))$ is said a regular (minimal) interval exchange set (linear complexity, neutrality, tree set$^1$, finite index basis, etc.).

Proposition
If $T$ is minimal, $w \in \mathcal{L}(T) \iff J_w \neq 0$.

Proposition
If $T$ is regular, $J_w$ is admissible for every $w \in \mathcal{L}(T)$.

1. See Valérie Berthé’s talk next week.
Return Theorem

The set of right return words to a word \( w \) (w.r.t. \( \mathcal{L}(T) \)) is

\[
\Gamma_{\mathcal{L}(T)} = \{ u \in \mathcal{L}(T) \mid wu \in A^+ \cap \mathcal{L}(T) \}
\]

while the set of first right return words is

\[
\mathcal{R}_{\mathcal{L}(T)} = \Gamma_{\mathcal{L}(T)} \setminus \Gamma_{\mathcal{L}(T)} A^+
\]

Theorem [BDDPRR (i.e. us), 2013]

Let \( T \) be a regular interval exchange transformation on \( A \). For any \( w \in \mathcal{L}(T) \), the set of first right return words to \( w \) is a basis of the free group on \( A \).
The set of first right return words to $a$ is

$$\mathcal{R}_{\mathcal{L}(T)}(a) = \{a, ba\}$$

And we can find it via the automorphism

$$\theta : \begin{cases} 
  a & \mapsto a \\
  b & \mapsto ba
\end{cases}$$
Return Theorem

The set of first right return words to $b$ is

$$R_{L}(T)(b) = \{aab, ab\}$$

And we can find it via the automorphism

$$\theta: \begin{cases} 
a &\mapsto ab &\mapsto aab 
b &\mapsto b &\mapsto ab
\end{cases}$$

Francesco Dolce (Université Paris Est)  Two-sided Rauzy Induction  Leiden, 23th January 2014
Conclusions

- Two-sided version of:
  - admissible semi-intervals,
  - Rauzy induction,
  - two Rauzy’s theorems;
- Regular interval exchange set;
- Return theorem.
Questions?