Quantum Knizhnik–Zamolodchikov equation: introduction and some applications

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The Knizhnik–Zamolodchikov equation is a set of compatible differential equations satisfied by the conformal blocks of the Wess–Zumino–Witten Conformal Field Theory, i.e., VEVs of (chiral) primary fields of the affine Lie algebra $\hat{g}$.

The compatibility condition (zero curvature condition) is related to a classical $r$-matrix.

The monodromy of these equations is related to the quantized algebra $U_q(\mathfrak{g})$. (NB: not the same $q$ as in the rest of this talk!)
The quantum Knizhnik–Zamolodchikov equation

Trying to extend this to quantized affine algebras $U_q(\hat{g})$ leads to the quantum Knizhnik–Zamolodchikov equation, a system of $q$-difference equations which historically appeared:

- in the study of form factors of integrable models [F. Smirnov, ’86]
- in the representation theory of quantum affine algebras [I. Frenkel, Reshetikhin ’92]
- in the study of correlation functions of integrable models [Jimbo, Miwa et al, ’93]
- in relation to representation theory of affine Hecke algebra and DAHA [Cherednik, Pasquier, ’90s]
A Cartan matrix (resp. affine Cartan matrix) is a square matrix $A = (a_{ij})$ of integers such that

- $a_{ii} = 2$.
- $a_{ij} \leq 0$ for $i \neq j$.
- $A$ is symmetrizable: there exists a diagonal matrix $D = \text{diag}(d_i)$, $d_i \in \mathbb{Z}_{>0}$, s.t. $DA$ symmetric ($d_i a_{ij} = d_j a_{ji}$) positive semi-definite.
- $A$ has maximal rank (resp. rank = maximal rank - 1).

Choose index range $I = \{1, \ldots, r\}$ (resp. $I = \{0, \ldots, r\}$) where $r$ is the rank. Choose the $d_i$ to be coprime.
Definition

The quantized algebra (resp. affine quantized algebra) associated to $A$ has generators $\{E_i, F_i, K_i, i \in I\}$, and relations

\[
K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i,
\]

\[
K_i E_j K_i^{-1} = q^{d_i a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-d_i a_{ij}} F_j, \quad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}
\]

\[
\sum_{k=0}^{1-a_{ij}} (-)^k \left[ \begin{array}{c} 1 - a_{ij} \\ k \end{array} \right] q^{d_i} (X_i)^{1-a_{ij}-k} X_j (X_i)^k = 0 \quad X \in \{E, F\}
\]

where we use the notation

\[
\left[ \begin{array}{c} m \\ n \end{array} \right]_q := \frac{(q^m - q^{-m}) \ldots (q^{m-n+1} - q^{-m+n-1})}{(q^n - q^{-n}) \ldots (q - q^{-1})}.
\]

We sometimes think of $K_i$ as $q^{H_i}$. 
Definition cont’d

Coalgebra structure and antipode:

\[
\begin{align*}
\Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, & S(E_i) &= -K_i^{-1}E_i, & \epsilon(E_i) &= 0 \\
\Delta(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i, & S(F_i) &= -F_i K_i, & \epsilon(F_i) &= 0 \\
\Delta(K_i) &= K_i \otimes K_i, & S(K_i) &= K_i^{-1}, & \epsilon(K_i) &= 1
\end{align*}
\]

Together, this forms a **Hopf algebra**.

In the affine case, since \( A \) has a one-dimensional kernel, there is a product of \( K_i^{\pm 1} \) which is central: call it \( q^\kappa \).

In what follows we stick to the affine untwisted case: we denote \( U_q(\mathfrak{g}) \) the quantized algebra associated to the submatrix \( (a_{ij})_{i,j=1,...,r} \), and \( U_q(\hat{\mathfrak{g}}) \) the affine quantized algebra.
Gradation

\[ U_q(\hat{\mathfrak{g}}) \] has a \( \mathbb{Z} \)-grading given by

\[ \deg E_i = \delta_{i,0}, \quad \deg F_i = -\delta_{i,0}, \quad \deg K_i = 0. \]

(choice of homogeneous gradation).

Equivalently, there is a one-parameter family of Hopf automorphisms

\[ \phi_z(x) = z^{\deg x} x, \quad z \in \mathbb{C}^\times \]

It is convenient to make these automorphisms inner by enlarging the algebra: define \( U_q(\tilde{\mathfrak{g}}) \) to have the same generators as \( U_q(\hat{\mathfrak{g}}) \), plus \( d \) (or \( z^d \)) such that

\[ [d, x] = (\deg x) x, \quad \phi_z(x) = z^d x z^{-d} \]

The Hopf algebra structure extends to \( U_q(\tilde{\mathfrak{g}}) \):

\[ \Delta(d) = 1 \otimes d + d \otimes 1, \quad S(d) = -d, \quad \epsilon(d) = 0. \]
Example: $U_q(\mathfrak{sl}(2))$

\[
A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad d_0 = d_1 = 1.
\]

Generators $E_1, F_1, K_1, E_0, F_0, K_0$ with usual relations including

\[
K_{1/0}^{-1} = q^{\pm 2} \left\{ \begin{array}{c} E_1 \\ F_0 \end{array} \right\}, \quad K_{1/0}^{-1} = q^{\mp 2} \left\{ \begin{array}{c} F_1 \\ E_0 \end{array} \right\}
\]

\[
X_i^3 X_j - [3] X_i^2 X_j X_i + [3] X_i X_j X_i^2 - X_j X_i^3 = 0 \quad X \in \{E, F\}
\]

where $[3] = q^{-2} + 1 + q^2$.

In particular, central element

\[
q^\kappa = K_0 K_1
\]
Universal R-matrix

\( U_q(\tilde{g}) \) possesses a universal R-matrix \( \mathcal{R} \in U_q(\tilde{g}) \otimes U_q(\tilde{g}) \), but only as a formal power series in \( q - 1 \). [Drinfeld, Khoroshkin–Tolstoy]

Formally, it satisfies

\[
\Delta'(x) = \mathcal{R} \Delta(x) \mathcal{R}^{-1} \]  \hspace{2cm} \text{(quasi-cocom)}

\[
(\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{23}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12} \]  \hspace{2cm} \text{(quasi-tri)}

(\text{where } \Delta'(x) = \sum x^{(2)} \otimes x^{(1)} \text{ if } \Delta(x) = \sum x^{(1)} \otimes x^{(2)} \text{ and commutes with grading.}
In general, quasi-commutativity allows to produce isomorphisms between representations $V \otimes W$ and $W \otimes V$:

$$\tilde{R}_{V,W} := P_{V,W}(\rho_V \otimes \rho_W)R$$

Here this isomorphism holds provided $\tilde{R}_{V,W}$ is well-defined.

One has

$$R = q^{\kappa \otimes d + d \otimes \kappa} R$$

where $R \in U_q(\hat{\mathfrak{g}}) \hat{\otimes} U_q(\hat{\mathfrak{g}})$. 
Finite-dimensional representations

- In any irreducible representation of $U_q(\hat{g})$, $q^\kappa$ acts as a scalar: call $q^\ell$ the eigenvalue, where $\ell$ is the level.
- Any finite-dimensional representation of $U_q(\hat{g})$ has level zero.
- Given a $U_q(\hat{g})$-module $V$, i.e., $\rho_V : U_q(\hat{g}) \to \mathfrak{gl}(V)$, and $z \in \mathbb{C}^\times$, we call $V(z)$ the module obtained by composing with the gradation automorphism: $\rho_V \circ \phi_z$.

Example: $U_q(\hat{\mathfrak{sl}(2)})$ spin 1/2 evaluation representation.

$V = \mathbb{C}^2$, action on $V(z)$:

$$E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

$$E_0 = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}, \quad F_0 = \begin{pmatrix} 0 & z^{-1} \\ 0 & 0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$$
We consider irreducible highest weight representations $W_\lambda$ of $U_q(\hat{g})$, i.e., there exists a cyclic vector $|\lambda\rangle$:

$$E_i |\lambda\rangle = 0, \quad K_i |\lambda\rangle = q^{\lambda_i} |\lambda\rangle, \quad i = 0, \ldots, r$$

They are obtained as the irreducible quotients of Verma modules. We assume the level generic, i.e., $q^\ell \not\in \mathbb{C} - q^\mathbb{N}$.

Define $U_q(\hat{p})$ to be the subalgebra of $U_q(\hat{g})$ generated by $E_i, F_i, i = 1, \ldots, r$ (i.e., $U_q(\hat{g})$), and $E_0$. Given an irreducible f.d. representation $W_\lambda$ of $U_q(\hat{g})$ (with highest weight $\lambda$) and a level $\ell$, we extend it to $U_q(\hat{p})$ by $E_0|_{W_\lambda} = 0, \quad q^{k}\lambda|_{W_\lambda} = q^{\ell}$, and then define a representation $W_\lambda$ of $U_q(\hat{g})$ by

$$W_\lambda = \text{Ind}_{U_q(\hat{p})}^{U_q(\hat{g})} W_\lambda \quad \lambda = (\lambda, \ell)$$

Remark: $W_\lambda$ graded ($U_q(\hat{g})$) module, with f.d. graded pieces.
We parameterize $\lambda = (\lambda_0 = \ell - 2s, \lambda_1 = 2s)$, where the spin $s \in \mathbb{N}/2$ and the level $\ell \not\in \mathbb{N}$. The highest weight vector $|s, \ell\rangle$ of $W_{s,\ell}$ satisfies

$$K_1 |s, \ell\rangle = q^{2s} |s, \ell\rangle, \quad K_0 |s, \ell\rangle = q^{\ell-2s} |s, \ell\rangle,$$

$$E_0 |s, \ell\rangle = E_1 |s, \ell\rangle = 0, \quad F_1^{2s+1} |s, \ell\rangle = 0$$
Intertwiners

We look for intertwiners

$$\Phi^V_{\mu \leftarrow \lambda}(z) : W_\lambda \rightarrow W_\mu \otimes V(z)$$

called type I Vertex Operators. Here $\lambda$ and $\mu$ must necessarily have the same level.

We also require compatibility with gradation:

$$x^d \Phi(z)x^{-d} = \Phi(x^{-1}z)$$

(not the standard CFT convention!) which fixes its normalization up to a constant.

Example: $U_q(\widehat{sl}(2))$, $V = \mathbb{C}^2$.

$$W_{s,\ell} \otimes V(z) \cong [s \geq 1/2] W_{s-1/2,\ell} \oplus W_{s+1/2,\ell}$$

so the nonzero intertwiners are the $\Phi^V_{s \leftarrow s \pm 1/2,\ell}(z)$. 
Expectation values of Vertex Operators

Given highest weights \((\lambda_0, \lambda_1, \ldots, \lambda_N)\) of same level, Define

\[
\Psi(z_1, \ldots, z_N) = \langle \lambda_0 | \Phi_{\lambda_0 \leftarrow \lambda_1}^{V_1}(z_1) \cdots \Phi(z_N)_{\lambda_{N-1} \leftarrow \lambda_N}^{V_N} | \lambda_N \rangle \\
\in V_1(z_1) \otimes \cdots \otimes V_N(z_N)
\]

where \(\langle \lambda_0 |\) is the lowest weight vector in the dual representation \(W_{\lambda_0}^{*}\).

\(\Psi\) is naively defined in a neighborhood of \(|z_1| \gg |z_2| \gg \cdots \gg |z_N|\), but we expect it to extend to a meromorphic function of the \(z_i\).

We want to show that \(\Psi\) satisfies a system of \(q\)-difference equations: [I. Frenkel, Reshetikhin]

\[
\Psi(z_1, \ldots, p z_i, \ldots, z_N) = S_i(z_1, \ldots, z_N)\Psi(z_1, \ldots, z_N)
\]

where \(p = q^{-2(\ell + \check{h})}\).

\(i = 1, \ldots, N\)
The graphical calculus is in essence common to most of mathematical physics (and consequently, topology), cf transfer matrices in stat mech and knot invariants. See [Kassel, XIV] for the case of nonaffine quantum groups.

A diagram

\[
\begin{array}{c}
V_1' \uparrow \hspace{1cm} V_2' \uparrow \hspace{1cm} \cdots \hspace{1cm} V_n' \uparrow \\
\vdots \hspace{2cm} \vdots \hspace{2cm} \cdots \hspace{2cm} \vdots \\
V_1 \downarrow \hspace{1cm} V_2 \downarrow \hspace{1cm} \cdots \hspace{1cm} V_m \downarrow \\
\end{array}
\]

corresponds to an element of

\[ \text{End}(V_1 \otimes V_2 \cdots \otimes V_m, V_1' \otimes V_2' \otimes \cdots \otimes V_n') \].
Here all vector spaces carry a $U_q(\hat{g})$ representation. We use thin lines for f.d. irreps, thick lines for highest weight irreps:

\[ V(z) \| z \| = W_\lambda \| \]

We are mostly interested in invariant objects, i.e., which commute with the quantum group action.
The R-matrix

We describe graphically the R-matrix using a crossing:

- When acting on a tensor product of two f.d. irreps:
  \[
  \hat{R}_{V_1(z_1), V_2(z_2)} = \frac{z_1}{z_2} = \hat{R}_+(z_1/z_2)
  \]

We also have

\[
\hat{R}^{-1}_{V_2(z_2), V_1(z_1)} = \frac{z_1}{z_2} = \hat{R}_-(z_1/z_2)
\]

Various relations become obvious, including the Yang–Baxter equation:
The R-matrix cont’d

- When acting on a tensor product of a f.d. irrep and a level $\ell$ highest weight irrep:

  \begin{align*}
  L_+(z) &= \begin{array}{c}
  \text{thick line} \\
  q^{\ell/2}z
  \end{array} \left/ \begin{array}{c}
  \text{thin line} \\
  q^{-\ell/2}z
  \end{array} \right. \\
  L^{-1}_+(z) &= \begin{array}{c}
  q^{\ell/2}z
  \end{array} \left/ \begin{array}{c}
  q^{-\ell/2}z
  \end{array} \right. \\
  L_-(z) &= \begin{array}{c}
  \text{thick line} \\
  q^{-\ell/2}z
  \end{array} \left/ \begin{array}{c}
  \text{thin line} \\
  q^{\ell/2}z
  \end{array} \right. \\
  L^{-1}_-(z) &= \begin{array}{c}
  q^{-\ell/2}z
  \end{array} \left/ \begin{array}{c}
  q^{\ell/2}z
  \end{array} \right.
  \end{align*}

Note that thin lines pick up a factor of $q^{\pm\ell}$ when they cross thick lines.
A type I Vertex Operator is depicted as:

$$
\Phi_{\mu \leftarrow \lambda}(z) = \frac{z}{W_\mu W_\lambda}
$$

and its VEV is:

$$
\Psi_{\lambda_0, \ldots, \lambda_N}(z_1, \ldots, z_N) = \frac{z_1}{W_{\lambda_0}} \frac{z_N}{W_{\lambda_N}} \frac{z}{W_{\lambda_1}} \frac{z}{W_{\lambda_{N-1}}}
$$

Remark: a type II VO would look like:

$$
\tilde{\Phi}(z) = \frac{z}{W_{\lambda}}
$$
A local relation

For non-graphical proofs, see [Frenkel, Reshetikhin] and [Idzumi, Tokihiro, Iohara, Jimbo, Miwa, Nakashima].

We start with a local relation:

\[ q^H = q \sum_{\alpha > 0} \alpha, \quad \text{and} \quad p = q^{-2(\ell+\hbar)}. \]

Proof: use \( S^2(x) = q^{-H+2\hbar d} x q^{H-2\hbar d} \) for all \( x \), i.e., rotating a line \( 2\pi \) clockwise amounts to inserting a \( q^{-H+2\hbar d} \).
A global relation

Now apply this relation to the correlation function $\Psi(z_1, \ldots, z_L)$ described graphically above, and use the highest/lowest weight property of $\langle \lambda_0 |$ and $| \lambda_N \rangle$:

$$\langle \lambda_0 | L_-(z) = \langle \lambda_0 | \otimes q^{\omega(\lambda_0)} \quad \text{and} \quad L_+(z) | \lambda_N \rangle = | \lambda_N \rangle \otimes q^{\omega(\lambda_N)}$$

$$q^{H + \omega(\lambda_0) + \omega(\lambda_N)}$$

which is the quantum Knizhnik–Zamolodchikov equation.
Remark on normalization of $R(z)$

Generically, if $V$ is a f.d. irrep, $V(z_1) \otimes V(z_2)$ is irreducible. This implies that $\tilde{R}_+(z)$ and $\tilde{R}_-(z)$ are proportional!

Let $R(z)$ be a normalization of the R-matrix (in general, distinct from $R_\pm(z)$) such that the unitarity equation holds:

$$\tilde{R}(z)\tilde{R}(1/z) = 1$$

We will depict it as

$$\tilde{R}(z_1/z_2) = \begin{array}{c} z_1 \\ \times \end{array} \begin{array}{c} z_2 \\ \times \end{array}$$

By a rescaling: $\psi \mapsto \prod_{i<j} f(z_i/z_j) \psi$ one can reduce $qKZ$ to its “flattened” version.
Affine Weyl group of type $\widehat{GL}(N)$

Consider diagrams on a cylindric strip made of $N$ lines connecting bottom vertices to top vertices:

Vertical concatenation gives them the group structure of extended affine Weyl group of type $\widehat{GL}(N)$, denoted $\hat{W}$. We shall show that $\hat{W}$ is generated by $R_i$, $i = 1, \ldots, N - 1$, and $P$:
There is a natural map from $\hat{W}$ to $W = S_N$ which to a diagram associates the permutation from bottom to top vertices:

The kernel of this map is a (normal) abelian subgroup isomorphic to $\mathbb{Z}^N$, with generators $S_i, i = 1, \ldots, N$:

$$S_i = \cdots$$

This gives the isomorphism $\hat{W} \cong W \rtimes \mathbb{Z}^N$, where a possible choice of $W \subset \hat{W}$ is the subgroup generated by $R_1, \ldots, R_{N-1}$. (here $\mathbb{Z}^N$ plays the role of coweight lattice of $GL(N)$)
Affine Weyl group of type $\hat{A}$

The affine Weyl group of type $\hat{A}_{N-1}$, i.e., $\widehat{SL}(N)$, is the Coxeter group with generators $R_i$, $i = 1, \ldots, N$, and relations

\begin{align*}
R_i R_{i+1} R_i &= R_{i+1} R_i R_{i+1} & & i \in \mathbb{Z}/N\mathbb{Z} \\
R_i R_j &= R_j R_i & & i \neq j \pm 1 \\
R_i^2 &= 1
\end{align*}

It is a subgroup of $\hat{W}$ (in fact, isomorphic to $W \rtimes \mathbb{Z}^{N-1}$). In particular, it does not contain $P$. 
Affine Weyl group of type \( \hat{A} \) cont’d

- \( \hat{W} \) is generated by \( R_1, \ldots, R_{N-1}, P \), since

\[
S_i = R_i \ldots R_{N-1} PR_1 \ldots R_{i-1}
\]

A possible (slightly redundant) set of relations of \( \hat{W} \):

\[
\begin{align*}
R_i R_{i+1} R_i &= R_{i+1} R_i R_{i+1} & i = 1, \ldots, N - 2 \\
R_i R_j &= R_j R_i & |i - j| > 1 \\
R_i^2 &= 1 & i = 1, \ldots, N - 1 \\
PR_i P^{-1} &= R_{i+1} & i = 1, \ldots, N - 2 \\
P^2 R_{N-1} P^{-2} &= R_1
\end{align*}
\]

- Note that \( P^N \) is central. The extended affine Weyl group of type \( \hat{A}_{N-1} \) is the quotient of \( \hat{W} \) by the relation \( P^N = 1 \). (in our framework, this extra relation is not so important)
Consider some functional space $\mathcal{F}$ of functions of $z_1, \ldots, z_N$, e.g., meromorphic functions on $(\mathbb{C}^\times)^N$. $\hat{\mathcal{W}}$ naturally acts on $\mathcal{F} \ni f$ by

\[
\begin{align*}
  r_i f(z_1, \ldots, z_N) &= f(z_1, \ldots, z_{i+1}, z_i, \ldots, z_N) & i = 1, \ldots, N - 1 \\
  p f(z_1, \ldots, z_N) &= f(z_2, \ldots, z_N, p^{-1} z_1)
\end{align*}
\]

In particular, note the action of the abelian subgroup:

\[
\begin{align*}
  s_i f(z_1, \ldots, z_N) &= f(z_1, \ldots, p^{-1} z_i, \ldots, z_N) & i = 1, \ldots, N
\end{align*}
\]
Given a f.d. irrep $V$ of $U_q(\hat{\mathfrak{g}})$, define $\mathcal{H} = V^\otimes N \otimes \mathcal{F}$, and

$$R_i = r_i \check{R}_i(z_i/z_{i+1}) \quad i = 1, \ldots, N - 1$$

$$P = p \rho$$

where $\rho$ is the twisted periodic shift of factors in $V^\otimes N$:

$$\rho(v_1 \otimes \cdots \otimes v_N) = v_2 \otimes \cdots \otimes v_N \otimes \Omega v_1$$

The $R_i$ form a representation of $W$ iff the R-matrix satisfies YBE and unitarity; and together with $P$, form a representation of $\hat{W}$ iff the twist $\Omega$ commutes with the R-matrix:

$$(\Omega \otimes \Omega) \check{R}(z) = \check{R}(z)(\Omega \otimes \Omega)$$

(e.g., is a group-like element of $U_q(\hat{\mathfrak{g}})$)
Reformulation of $qKZ$

As the graphical formulation suggests, the $qKZ$ equation is nothing but

$$S_i \psi = \psi \quad i = 1, \ldots, N$$

(for some specific choice of $\Omega$), i.e., the statement that $\psi$ is invariant under the normal abelian subgroup $\mathbb{Z}^N$.

In other words, $S_i = s_i S_i(z_1, \ldots, z_N)$. 
qKZ in arbitrary type

More generally, fix an extended affine Weyl group, i.e.,

$$\hat{W} \cong W \times \Lambda$$

where $\Lambda$ is the lattice of coweights of the (finite-dimensional) Lie algebra with Weyl group $W$.

Remark: such affine Weyl groups/coweight lattices are classified by affine Dynkin diagrams (i.e., affine Cartan matrices), but they’re unrelated to $\hat{g}$.

It acts naturally as reflections/translations on the coweight space $\mathbb{C}^N \supset \Lambda$: $x_i \mapsto A(w)_{ij}x_j + B(w)_i$; with $A(w)_{ij} \in \{-1, 0, 1\}$; and therefore on $\mathcal{F}$ in multiplicative form:

$$r_w z_i = p^{B(w^{-1})_i} \prod_{j=1}^{N} z_j^{A(w^{-1})_{ij}}$$
Given a (f.d.) vector space $\mathcal{V}$, define **generalized R-matrices** to be operators $\check{R}_w \in \text{End}(\mathcal{V}) \otimes \mathcal{F}$, $w \in \hat{W}$, such that the operators $R_w = r_w \check{R}_w$ form a representation of $\hat{W}$ on $\mathcal{V} \otimes \mathcal{F}$.

In practice, one only needs to specify the $\check{R}_w$ for $w$ in some generating set of $\hat{W}$.

The $q$KZ equation then reads

$$S \psi = \psi \quad \forall S \in \Lambda$$

i.e., invariance under the normal abelian subgroup $\Lambda$. 
Example: \( \hat{C}_N = \begin{array}{cccc}
0 & 1 & \cdots & N-1 & N \\
\end{array} \)

Elements of \( \hat{W} \) are diagrams on a strip with “mirrors” on the two sides:

The abelian subgroup is generated by elements of the type:
The nodes $i = 1, \ldots, N - 1$ act by exchange of $z_i$ and $z_{i+1}$, whereas $i = 0$ is $z_1 \to p^{-1} z_1^{-1}$, and $i = N$ is $z_N \to z_N^{-1}$.

The R-matrices in the bulk $\check{R}_i = \begin{array}{c} z_i \\ z_{i+1} \end{array}$, $i = 1, \ldots, N - 1$, satisfy the ordinary YBE; at the right boundary, $\check{R}_N \equiv K = \begin{array}{c} z_N^{-1} \\ z_N \end{array}$ and $\check{R}_{N-1}$ satisfy the reflection (or boundary Yang–Baxter) equation:

and similarly at the left boundary.
Given a “full” set of solutions $\Psi = (\Psi_a)_{a=1,...,d}$, because of the semi-direct product structure of $\hat{W}$, we have an action of $W$ on the space of solutions, i.e., $\forall i \in I, a, R_i\Psi_a$ is still a solution ($I$ generating set of $W$). This implies

$$R_i\Psi = \Psi M_i$$

but $M_i = (M_i)_{a,b}$ is not necessarily constant! can be $p$-periodic function of spectral parameters (really, only of $z_i/z_{i+1}$). $R_i^2 = 1$ implies

$$\Psi = (R_i)^2\Psi = R_i(\Psi M_i) = (R_i\Psi)(r_iM_i) = \Psi M_ir_iM_i$$

i.e., $M_i$ satisfies unitarity. Braid relation implies YBE for $M_i$...
Interpretation of $M_i$ for VOs

In the case of type $\hat{A}$, the $M_i$ are known to be dynamical elliptic solutions of YBE. (similar results hold in type $\hat{C}$)

A graphical interpretation in type $\hat{A}$: if $\Psi$ is the collection of VEVs of VOs corresponding to different $\lambda$’s, then the connection formula corresponds to commutation of two VOs:

\[
\begin{align*}
\lambda_0 & \quad \lambda_1 & \quad \lambda_2 \\
\downarrow & \quad \downarrow & \quad \downarrow \\
\lambda_1 & \quad \lambda_2 & \quad \lambda_0
\end{align*}
\]

\[
\begin{align*}
\lambda_0 & \quad \lambda_1 & \quad \lambda_2 \\
\downarrow & \quad \downarrow & \quad \downarrow \\
\lambda_1 & \quad \lambda_2 & \quad \lambda_0
\end{align*}
\]
Reduced $q$KZ equation

One may look for $\Psi$ which are invariant under the whole of $\hat{W}$, i.e., that are $W$-invariant as well.

For example, in type $\hat{A}$, the equations look like

$$\hat{R}_i(z_i/z_{i+1})\Psi = r_i\Psi = \Psi(\ldots, z_{i+1}, z_i, \ldots)$$

$$\rho \Psi = p^{-1}\Psi = \Psi(z_2, \ldots, z_N, p z_1)$$

Remark: by an abstract of theorem of Cherednik, or (at least in type $\hat{A}$) by use of a factorizing twist for $M_i$, one can turn a full set of solution $\Psi = (\Psi_a)$ into a full set of $W$-invariant solutions $\tilde{\Psi} = \Psi G$. 
Lecture not given: consult the book
Jimbo, M. and Miwa, T.
Algebraic analysis of solvable lattice models
Semi-classical limit

One can look at the “quasi-classical limit” of $q$KZ where $p \to 1$.

NB: this is not necessarily the same as $q \to 1$!

Set $p = e^{\hbar}$. Then [Reshetikhin, ’93]

$$\Psi = e^{F/\hbar}\Psi_0(1 + O(\hbar))$$

where $F$ is a scalar function.

Expanding the $q$KZ equation $S_i\Psi = \Psi(pz_i)$ at first order in $\hbar$, one finds:

$$S_i(p = 1)\Psi_0 = \exp \left( z_i \frac{\partial}{\partial z_i} F \right) \Psi_0$$

i.e., $\Psi_0$ is eigenvector of $S_i = S_i(p = 1)$ with eigenvalue $\exp \left( z_i \frac{\partial}{\partial z_i} F \right)$. 

We now focus on type $\hat{A}$ (though other types work as well). At $p = 1$, $S_i$ commutes with the (twisted, inhomogeneous) transfer matrix

$$T(z) = z_1 \quad z_2 \quad \cdots \quad z_N \quad \Omega$$

So we expect that $\Psi_0$ is also an eigenvector of the transfer matrix.

In general, solving $qKZ$ is harder than diagonalizing the transfer matrix! Except...
Consider for example $U_q(\widehat{\text{sl}(2)})$ VOs. For generic $\ell$, the $q$-conformal blocks look like

$$\Psi = \langle s_0 | \Phi^{s_0,s_1}(z_1) \cdots \Phi^{s_{N-1},s_N}(z_N) | s_N \rangle$$

When the level is a positive integer, the paths are truncated:

When $\ell = 2$

$$(\text{spin } 1/2, s_0 = s_N = 0)$$
Perfect VOs

If $\ell = 1, 2, \ldots$ ($p = q^{-2(\ell+2)}$), and $\Phi(z)$ is a so-called “perfect vertex operator” (i.e., for $U_q(\mathfrak{sl}(2))$, of spin $\ell/2$), then

- There is only one conformal block (single path).

- $\Psi$ satisfies the reduced $qKZ$ equation.

- $\Psi$ has a simple analytic behavior – a product of two-body factors times a vector-valued polynomial in $z_1, \ldots, z_L, q, q^{-1}$.

- $\Psi$ has a smooth limit as $p \to 1$.

- If $q \to -e^{-i\pi/(\ell+2)}$ as $p \to 1$, then $\Psi_0 = \lim_{p \to 1} \Psi$ is the ground state eigenvector of the (twisted, inhomogeneous) transfer matrix $T(z)$ (in some regions of the $z_i$ and $z$, including the homogeneous limit $z_i \to 1$).
Construction of VOs at integer level

General idea: use \((q\text{-deformed})\) bosonization technique.

Here we have a \((q\text{-deformed})\) chiral WZW model at level \(\ell\).

\[
c = \frac{3\ell}{\ell + 2}
\]

→ use a boson plus a parafermion. [Fonseca, PZJ, ’13]

\[
c = 1 + \frac{2(\ell - 1)}{\ell + 2}
\]

Remark: other bosonization techniques exist (Wakimoto).
Example: $XXZ \, \Delta = -1/2$

For $\ell = 1$ [Razumov, Stroganov, PZJ ’07], we obtain this way the exact finite-size ground state of the six-vertex model at $\Delta = \frac{1}{2}(q + q^{-1}) = -1/2$, or in the limit $z_i \to 1$, of the XXZ spin chain at $\Delta = -1/2$:

$$H = -\frac{1}{2} \sum_{i=1}^{N} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z \right)$$

where the $\sigma_{i}^{x,y,z}$ are Pauli matrices acting on the $i^{th}$ factor of $(\mathbb{C}^2)^\otimes N$. (twisted PBC/PBC in even/odd size)

- There are explicit integral formulae for the entries of $\Psi$.
- They provide deep connections with combinatorics. For example, in odd size $N = 2n + 1$, $\Psi_{max} = A_n$ (number of Alternating Sign Matrices of size $n$).
- A lot more is known at $\ell = 1$, e.g., the exact formula for the Emptiness Formation Probability [Cantini ’11].
Lecture not given. consult e.g. Rimányi, R. and Tarasov, V. and Varchenko, A. and Zinn-Justin, P.
Extended Joseph polynomials, quantized conformal blocks, and a $q$-Selberg type integral