Spatial Epidemics: Critical Behavior

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Mean Field Models
  Stochastic Logistic (SIS) Model
  Reed-Frost (SIR) Model
  Branching Envelopes
  Critical Behavior

Spatial Epidemic Models
  Spatial SIS and SIR Models
  Branching Random Walks and Superprocess Limits
  Spatial Epidemic Models: Critical Scaling
  Spatial Extent of SuperBM
Stochastic Logistic (SIS) Model

- Population Size $N < \infty$
- Individuals susceptible (S) or infected (I).
- Infecteds recover in time 1, then immune for one time unit.
- Infecteds infect susceptibles with probability $p$. 
Stochastic Logistic (SIS) Model

- Population Size $N < \infty$
- Individuals susceptible (S) or infected (I).
- Infecteds recover in time 1, then immune for one time unit.
- Infecteds infect susceptibles with probability $p$.

Dynamics:

\[
S_{t+1} = N - I_{t+1},
\]
\[
I_{t+1} = Y_{t,1} + Y_{t,2} + \cdots + Y_{t,S_t};
\]
\[
Y_{t,j} \sim \text{Bernoulli}(1 - p^t).
\]
Stochastic Logistic (SIS) Model: Variation

- Population Size $N < \infty$
- Individuals susceptible (S) or infected (I).
- Infecteds recover in time 1, then immediately susceptible.
- Infecteds infect susceptibles with probability $p$.

Dynamics:

\[
S_{t+1} = N - I_{t+1}, \\
I_{t+1} = Y_{t,1} + Y_{t,2} + \cdots + Y_{t,N}; \\
Y_{t,j} \sim \text{Bernoulli-}(1 - p^t).
\]
Reed-Frost (SIR) Model

- Population Size $N < \infty$
- Individuals susceptible (S), infected (I), or recovered (R).
- Recovered individuals immune from further infection.
- Infecteds recover in time 1.
- Infecteds infect susceptibles with probability $p$.

Dynamics:

$$S_{t+1} = S_t - I_{t+1},$$
$$R_{t+1} = R_t + I_t,$$
$$I_{t+1} = Y_{t,1} + Y_{t,2} + \cdots + Y_{t,S_t},$$
$$Y_{t,j} \sim \text{Bernoulli}(1 - p^t).$$
Recovered individuals are immune from future infection.
Reed-Frost and Random Graphs

Reed-Frost model is equivalent to the Erdős-Renyi random graph model:

Individuals $\leftrightarrow$ Vertices
Infections $\leftrightarrow$ Edges
Epidemic $\leftrightarrow$ Connected Components
Branching Envelope of an Epidemic Model

Each epidemic model has a branching envelope:

- When $I_t \ll S_t$, infected set grows $\approx$ branching process.
- Mean-field models :: Galton-Watson processes.
- Stratified models :: Multitype Galton-Watson processes.
- Spatial models :: Branching Random Walks.
- RF and SL epidemics: offspring distribution Binomial-$\left( N, p \right)$.
- **Coupling**: Epidemics dominated by branching envelopes.

**Criticality**: Mean Offspring $= 1 \iff p = 1/N$. 
Example: SIS Epidemic and its Branching Envelope

\[
N = 80000 \\
l_0 = 200 \\
p = 1/80000
\]
Feller’s Theorem

- $Z_t^N$: Galton-Watson processes
- Offspring Distribution $F$: mean 1, variance $< \infty$.
- Initial Conditions: $Z_0^N = aN$.

Then

$$Z_{Nt}/N \rightarrow Y_t$$

where $Y_t$ is a Feller diffusion process:

$$Y_0 = a$$

$$dY_t = \sqrt{Y_t} \, dB_t$$
Feller’s Theorem

- $Z^N_t$: Galton-Watson processes
- Offspring Distribution $F$: mean $1 - b/N$, variance $< \infty$.
- Initial Conditions: $Z^N_0 = aN$.

Then

$$Z^N_{Nt}/N \rightarrow Y_t$$

where $Y_t$ is a Feller diffusion process with drift:

$$Y_0 = a$$

$$dY_t = \sqrt{Y_t} dB_t - bY_t dt$$
Critical Heuristics: SIS Epidemics

- Critical Epidemic with $I_0 = m$ should last $\approx m$ generations.
- Offspring in branching envelope :: attempted infections.
- Collisions: Duplicate infections not allowed.
- Critical Threshold: $\#$ collisions/generations $\approx O(1)$
Critical Heuristics: SIS Epidemics

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- Offspring in branching envelope :: attempted infections.
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Critical SIS Epidemic:

$$E(\# \text{collisions in generation } t + 1) \approx \frac{l_t^2}{N}$$

so observable deviation from branching envelope when $l_t \approx \sqrt{N}$. 
Critical Heuristics: SIR Epidemics

- Critical Epidemic with $I_0 = m$ should last $\approx m$ generations.
- Offspring in branching envelope $::$ attempted infections.
- Collisions: Infections of immunes $::$ not allowed.
- Critical Threshold: $\#$ collisions/generations $\approx O(1)$

Critical SIR Epidemic:

$$E(\#\text{collisions in generation } t + 1) \approx I_t(N - S_t)/N$$

so observable deviation from branching envelope when $I_t \approx N^{1/3}$.  

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Critical Behavior: SIS Epidemics

- Population size: \( N \to \infty \)
- # Infected in Generation \( t \): \( I_t^N \)
- Initial Condition: \( I_0^N \sim bN^\alpha \).

Theorem: \( I_{N^\alpha t}/N^\alpha \to Y_t \) where

\[
Y_0 = b; \\
dY_t = \sqrt{Y_t} \ dB_t \quad \text{if} \quad \alpha < 1/2 \\
dY_t = \sqrt{Y_t} \ dB_t - Y_t^2 \ dt \quad \text{if} \quad \alpha/2
\]

Note: When \( \alpha = 1/2 \) the initial condition \( b = \infty \) is permitted, as \( \infty \) is an entrance boundary for the limit diffusion.
Critical Behavior: SIS Epidemics

- Population size: \( N \to \infty \)
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\[
dY_t = \sqrt{Y_t} \, dB_t - Y_t^2 \, dt \quad \text{if} \quad \alpha / 2
\]

**Note:** When \( \alpha = 1/2 \) the initial condition \( b = \infty \) is permitted, as \( \infty \) is an entrance boundary for the limit diffusion.
Example: Entrance Boundary

\[ N = 80000 \]
\[ I_0 = 10000 \]
\[ p = 1/80000 \]
Critical Behavior: SIS Epidemics

- Population size: \( N \to \infty \)
- # Infected in Generation \( t \): \( I_t^N \)
- Initial Condition: \( I_0^N \sim bN^\alpha \).

**Corollary:** If \( \alpha = 2 \) then

\[
\sum_{t \geq 0} \frac{I_t^N}{N} \to \tau(b)
\]

where \( \tau(b) = \) first passage time to zero of Ornstein-Uhlenbeck process started at \( b \).
Critical Behavior: Reed-Frost (SIR) Epidemics

- Population size: $N \to \infty$
- # Infected in Generation $t$:$= I_t^N$
- # Recovered in Generation $t$: $= R_t^N$
- Initial Condition: $I_0^N \sim bN^\alpha$

Theorem:

$$\left( \begin{array}{c} N^{-\alpha} I_t^N \\ N^{-2\alpha} R_t^N \end{array} \right) \xrightarrow{D} \left( \begin{array}{c} I(t) \\ R(t) \end{array} \right)$$

The limit process satisfies $I(0) = b$ and

$$dR(t) = I(t) \, dt$$

$$dl(t) = +\sqrt{l(t)} \, dB_t \quad \text{if } \alpha < 1/3$$

$$dl(t) = +\sqrt{l(t)} \, dB_t - I(t)R(t) \, dt \quad \text{if } \alpha = 1/3$$
Critical Behavior: Reed-Frost (SIR) Epidemics

- Population size: $N \to \infty$
- # Infected in Generation $t$: $I_t^N$
- # Recovered in Generation $t$: $R_t^N$
- Initial Condition: $I_0^N \sim bN^\alpha$

**Corollary (Martin-Löf; Aldous)** If $\alpha = 1/3$ then

$$R_\infty^N / N^{2/3} \to \tau(b)$$

where $\tau(b)$ = first passage time of $B(t) + t^2/2$ to $b$. 
Spatial SIS and SIR Epidemics

- Villages $V_x$ at Sites $x \in \mathbb{Z}$
- Village Size: $= N$
- Nearest Neighbor Disease Propagation
- Reed-Frost (SIR) or Stochastic Logistic (SIS) Rules Locally.
Spatial SIS and SIR Epidemics

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- Village Size: $N$
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Random Graph Formulation:

- SIR Epidemic $\iff$ Percolation on $\mathbb{Z} \times K_N$
- SIS Epidemic $\iff$ Oriented Percolation on $\mathbb{Z}^2 \times K_N$. 
Spatial SIS and SIR Epidemics

- Villages $V_x$ at Sites $x \in \mathbb{Z}$
- Village Size: $= N$
- Nearest Neighbor Disease Propagation
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Associated Measure-Valued Processes

$$X_t^M = X_t^{M,N}: \text{measure that puts mass } 1/M \text{ at } x/\sqrt{M} \text{ for each particle at site } x \text{ at time } t.$$
Branching Envelope of a Spatial Epidemic

Nearest Neighbor Branching Random Walk:

- Particle at $x$ puts offspring at $x - 1$, $x$, $x + 1$
- #Offspring are independent Binomial $-(N, p)$
Branching Envelope of a Spatial Epidemic

Nearest Neighbor Branching Random Walk:

- Particle at $x$ puts offspring at $x - 1, x, x + 1$
- #Offspring are independent Binomial $- (N, p)$

Associated Measure-Valued Processes

$$X_t^M = X_t^{M,N}:$$

measure that puts mass $1/M$ at $x/\sqrt{M}$ for each particle at site $x$ at time $t$. 
Watanabe’s Theorem I

Let $X^M_t$ be the measure-valued process associated to a critical nearest neighbor branching random walk. If

$$X^M_0 \Longrightarrow X_0$$

then

$$X^M_{Mt} \Longrightarrow X_t$$

where $X_t$ is the Dawson-Watanabe process (superBM). The DW process solves the martingale problem

$$E\langle X_t, \phi \rangle - E\langle X_0, \phi \rangle = CE \int_0^t \langle X_s, \Delta \phi \rangle \, ds \quad \forall \, \phi \in C_c^2(\mathbb{R})$$
Watanabe’s Theorem II

Let \( X_t^M \) be the measure-valued process associated to a critical nearest neighbor branching random walk with particles killed at rate \( a/M \). If

\[
X_0^M \xrightarrow{\text{d}} X_0
\]

then

\[
X_{Mt}^M \xrightarrow{\text{d}} X_t
\]

where \( X_t \) is the Dawson-Watanabe process with killing rate \( a \). This process solves the martingale problem

\[
E\langle X_t, \phi \rangle - E\langle X_0, \phi \rangle = E \int_0^t \langle X_s, C \Delta \phi - a \phi \rangle \, ds \quad \forall \, \phi \in C_c^2(\mathbb{R})
\]
Superposition Principle

Corollary: Let $X_t^{\mu}$ be a Dawson-Watanabe process with killing rate $a$ and initial state $X_0^{\mu} = \mu$.

If $X^{\mu}$ and $X^{\nu}$ are independent Dawson-Watanabe processes with initial conditions $\mu$ and $\nu$ then

$$X_t^{\mu} \cup X_t^{\nu} \overset{D}{=} X_t^{\mu+\nu}$$
Branching Random Walk: Simulation
Branching Random Walk: Local Behavior

\[ Y_n(x) := \text{# particles at site } x \in \mathbb{Z} \text{ in } n\text{th generation of critical branching random walk} \]
Branching Random Walk: Local Behavior

\[ Y_n(x) := \text{# particles at site } x \in \mathbb{Z} \text{ in } n\text{th generation of critical branching random walk} \]

Theorem: \( \forall \varepsilon > 0 \text{ and } \forall L < \infty \ \exists \delta = \delta(\varepsilon, L) > 0 \) such that

\[
\sum_{x \in \mathbb{Z}} Y_0(x) \leq Ln \quad \text{and} \quad \sum_{x:|x|>L\sqrt{n}} Y_0(x) = 0 \implies
\]

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Spatial Epidemics: Critical Behavior
Branching Random Walk: Local Behavior

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**Theorem:** \( \forall \varepsilon > 0 \) and \( \forall L < \infty \) \( \exists \delta = \delta(\varepsilon, L) > 0 \) such that

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\sum_{x \in \mathbb{Z}} Y_0(x) \leq Ln \quad \text{and} \quad \sum_{x:|x|>L\sqrt{n}} Y_0(x) = 0 \implies
\]

\[
\max_{x \in \mathbb{Z}} P \left\{ \left| Y_n(x) - \frac{1}{2\delta \sqrt{n}} \sum_{y:|y-x|\leq \delta \sqrt{n}} Y_n(y) \right| \geq \varepsilon \sqrt{n} \right\} < \varepsilon.
\]
Scaling Limits: SIS Spatial Epidemics

**Theorem:** Let $X_t^N = X_t^{M,N}$ be the measure-valued process associated with critical SIS spatial epidemic with village size $N$ and scaling $M = N^\alpha$. If $X_0^N \Rightarrow X_0$ then

$$X_{Mt}^N \Rightarrow X_t$$

where

- If $\alpha < 2/3$ then $X_t$ is the Dawson-Watanabe process.
- If $\alpha = 2/3$ then $X_t$ is the Dawson-Watanabe process with killing rate $X(t, x)^2$. 
Scaling Limits: SIS Spatial Epidemics

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- If $\alpha < 2/3$ then $X_t$ is the Dawson-Watanabe process.
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**Note 1:** One-dimensional Dawson-Watanabe process has continuous density $X(t, x)$. 

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Scaling Limits: SIS Spatial Epidemics

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where

- If $\alpha < 2/3$ then $X_t$ is the Dawson-Watanabe process.
- If $\alpha = 2/3$ then $X_t$ is the Dawson-Watanabe process with killing rate $X(t, x)^2$.

Note 2: MG problem for DW with killing rate $\theta(t, x)$:

$$E\langle X_t, \phi \rangle = E\langle X_0, \phi \rangle + E \int_0^t \langle X_t, C \Delta \phi - \theta(t, \cdot) \phi \rangle \, ds$$
Scaling Limits: SIR Spatial Epidemics

Theorem: Let \( X_t^N = X_t^{M,N} \) be the measure-valued process associated with critical SIR spatial epidemic with village size \( N \) and scaling \( M = N^\alpha \). If \( X_0^N \Rightarrow X_0 \) then

\[
X_{Mt}^N \Longrightarrow X_t
\]

where

- If \( \alpha < 2/5 \) then \( X_t \) is the Dawson-Watanabe process.
- If \( \alpha = 2/5 \) then \( X_t \) is the Dawson-Watanabe process with killing rate

\[
X(t, x) \int_0^t X(s, x) \, ds
\]
Critical Scaling: Heuristics (SIS Epidemics)

- # Infected Per Generation: $\approx M$
- Duration: $\approx M$ generations.
- # Infected Per Site: $\approx \sqrt{M}$
- # Collisions Per Site: $\approx M/N$
- # Collisions Per Generation: $\approx M^{3/2}/N$

So if $M \approx N^{2/3}$ then # Collisions Per Generation $\approx 1$. 
Critical Scaling: Heuristics (SIR Epidemics)

- # Infected Per Generation: $\approx M$
- Duration: $\approx M$ generations.
- # Infected Per Site: $\approx \sqrt{M}$
- # Recovered Per Site: $\approx M\sqrt{M}$
- # Collisions Per Site: $\approx M^2/N$
- # Collisions Per Generation: $\approx M^{5/2}/N$

So if $M \approx N^{2/5}$ then # Collisions Per Generation $\approx 1$. 
Critical Spatial SIS Epidemic: Simulation
Higher Dimensions

\[ d \geq 3: \]

- In critical branching random walk with \( N \) particles, after \( \varepsilon N \) generations particles distribute themselves \( O_P(1) \) per occupied site.
- In critical SIS epidemic with village size \( N \), collision rate \( O_P(1/N) \) per site.
- So the critical threshold is at \( N \).
- Scaling Limit: Dawson-Watanabe with constant killing rate.
Higher Dimensions

Critical Dimension: $d = 2$

- In critical branching random walk with $N$ particles, after $\varepsilon N$ generations particles distribute themselves $O_P(\log N)$ per occupied site.
- In critical SIS epidemic with village size $N$, collision rate $O_P(\log N/N)$ per site.
- So the critical threshold is at $N/\log N$. 

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Spatial Epidemics: Critical Behavior
Long-Range Contact Process: Critical Scaling

- Sites $x \in \mathbb{Z}/N^2$
- Status of site $x$ at time $t$: $\xi^N_t(x) = 0$ or $1$ (healthy/infected)
- Dynamics:
  - Infected sites become healthy at rate 1
  - Healthy site $x$ becomes infected at rate $NX^N(t, x)$ where

$$X^N(t, x) := \frac{1}{\sqrt{N}} \sum_{y:|y-x| \leq 1/\sqrt{N}} \xi^N_t(x)$$

Theorem (Mueller & Tribe): $X^N(t, x) \Rightarrow X_t$ where $X_t$ is a Dawson-Watanabe process with killing rate $CX^2(t, x)$. 
Long-Range Contact Process: Critical Scaling

- Sites $x \in \mathbb{Z}/N^2$
- Status of site $x$ at time $t$: $\xi^N_t(x) = 0$ or $1$ (healthy/infected)
- Dynamics:
  - Infected sites become healthy at rate $1$
  - Healthy site $x$ becomes infected at rate $NX^N(t, x)$ where

$$X^N(t, x) := \frac{1}{\sqrt{N}} \sum_{y: |y-x| \leq 1/\sqrt{N}} \xi^N_t(x)$$

**Theorem (Mueller & Tribe):** $X^N(t, x)dx \Rightarrow X_t$ where $X_t$ is a Dawson-Watanabe process with killing rate $CX(t, x)^2$. 
Spatial Extent of Dawson-Watanabe Process

- $X_t =$ Dawson-Watanabe process
- $\mathcal{R}(X) := \bigcup_{t \geq 0} \text{support}(X_t)$
- $u_D(x) := -\log P(\mathcal{R}(X) \subset D \mid X_0 = \delta_x)$
Spatial Extent of Dawson-Watanabe Process

- $X_t =$ Dawson-Watanabe process
- $\mathcal{R}(X) := \bigcup_{t \geq 0} \text{support}(X_t)$
- $u_D(x) := - \log P(\mathcal{R}(X) \subset D \mid X_0 = \delta_x)$

**Theorem (Dynkin):** For any finite interval $D$, $u_D(x)$ is the maximal nonnegative solution in $D$ of the differential equation

$$u'' = u^2$$
Spatial Extent of Dawson-Watanabe Process

- $X_t =$ Dawson-Watanabe process
- $\mathcal{R}(X) := \bigcup_{t \geq 0} \text{support}(X_t)$
- $u_D(x) := -\log P(\mathcal{R}(X) \subset D \mid X_0 = \delta_x)$

**Solution:** Weierstrass $\mathcal{P} -$ Function

\[
u_D(x) = \mathcal{P}_L(x/\sqrt{6}) = \frac{1}{6x^2} + \sum_{\omega \in L^*} \left\{ \frac{1}{6(x - \omega)^2} - \frac{1}{\omega^2} \right\}
\]

where the **period lattice** $L$ is generated by $Ce^{\pi i/3}$ for $C > 0$ depending on $D = [0, a]$ as follows:

\[C = \sqrt{6}a\]
Spatial Extent of Dawson-Watanabe Process

- $X_t = \text{Dawson-Watanabe process}$
- $\mathcal{R}(X) := \bigcup_{t \geq 0} \text{support}(X_t)$
- $u_D(x) := -\log P(\mathcal{R}(X) \subset D \mid X_0 = \delta_x)$

**General Initial Conditions:** For any finite Borel measure $\mu$ with support $\subset D$, 

$$- \log P(\mathcal{R}(X) \subset D \mid X_0 = \mu) = \int u_D(x) \mu(dx)$$

$$= \int \mathcal{P}_L(x/\sqrt{6}) \mu(dx)$$