LECTURES ON ARITHMETIC DIFFERENTIAL EQUATIONS

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1. Classical analogies between numbers and functions

functions	numbers
$\mathbb{C}[x]$	$\mathbb{Z}$
finite extensions $K/\mathbb{C}(x)$	finite extensions $K/\mathbb{Q}$
compact Riemann surface S	$\overline{Spec\ \mathcal{O}_K}$
points on S	places of K
Jacobian variety $Cl^0(S)$	divisor class group $Cl(K)$
unramified finite abelian covers $S' \to S$ and link with $Cl^0(S)$ via $H_1(S,\mathbb{Z})$	unramified finite abelian extensions K'/K and link with $Cl(K)$ (Hilbert class field)
ramified version of this	class field theory, ray class groups
non-abelian version of this $(\pi_1(S\backslash\{P_1,...,P_n\}))$	non-abelian class field theory $(G(K^a/K)$ and Langlands)
Riemann-Roch	arithmetic analogue of Riemann-Roch
differential calculus and differential equations on S	NO CLASSICAL ANSWER FOR A POSSIBLE ANSWER SEE NEXT SECTIONS
$\mathbb{C}[x_1,x_2]$	NO CLASSICAL ANSWER MAYBE $\mathbb{Z}\otimes_{\mathbb{F}_1}\mathbb{Z}$ FOR A HYPOTHETICAL $\mathbb{F}_1$???) FOR AN ALTERNATIVE NEXT SECTIONS

2. ODEs, Part I

ordinary diff equations satisfied by functions (differential algebra)	ordinary "diff" equations satisfied by numbers (arithmetic differential algebra)
main reference for material below AB, "Differential algebra and Diophantine geometry", Hermann 1994	main reference for material below: AB, "Arithmetic differential equations" Math Surv Mono, AMS 2005 (this theory is $\perp$ to Dwork's p-adic differential equations where diff equations are satisfied by functions)
Applications Geometric Lang Conjecture (AB, Annals 92) Effective bound for GLC (AB, Duke 93) Effective geo Manin-Mumford (AB, Duke 94)	Applications Arithmetic Th of the kernel (Invent 95) Effective Manin-Mumford (AB, Duke 96) Modular forms (AB, Crelle 2000) Heegner points (AB+Poonen, Compositio 2009)
R a ring $\delta : R \to R$ is a derivation if $\delta(x+y) = \delta x + \delta y$ $\delta(xy) = x\delta y + y\delta x$ (Ritt, Kolchin)	R a ring, $p \in \mathbb{Z}$ a prime, non-zero divisor in R $\delta : R \to R$ is a p-derivation if $\delta 1 = 0$ and $\delta(x+y) = \delta x + \delta y - \frac{(x+y)^p - x^p - y^p}{p}$ $\delta(xy) = x^p \delta y + y^p \delta x + p\delta x \delta y$ (Joyal C.R Acad Sci Canada 85, AB Invent 95) for theory below: AB Invent 95, Duke 96

δ is a derivation iff $R \to D_2(R) = (R \times R, \text{dual number structure})$ $x \mapsto (x, \delta x)$ is a ring homomorphism	δ is a p-derivation iff $R \to W_2(R) = (R \times R, \text{Witt vector structure})$ $x \mapsto (x, \delta x)$ is a ring homomorphism.
	δ is a p-derivation iff $\phi : R \to R$ $\phi(x) = x^p + p\delta x$ is a ring homomorphism so p-derivations $R \to R$ are in bijection with ring homomorphisms $\phi : R \to R$ lifting Frobenius $R/pR \to R/pR$.
$R^\delta = \{x \in R; \delta x = 0\}$ subring of R	$R^\delta = \{x \in R; \delta x = 0\}$ submonoid of R
Examples 1) $R = \mathbb{C}[x]$, $\delta = d/dx$ $R^\delta = \mathbb{C}$ 2) $R = C^\infty(N, \mathbb{C})$, $N = \mathbb{R}, \mathbb{R}/\mathbb{Z}$ $\delta = d/dx$ $R^\delta = \mathbb{C}$	*Examples* 1) $R = \mathbb{Z}$, $\delta x = \frac{x - x^p}{p}$ (unique) $R^\delta = \{-1, 0, 1\}$ 2) $R = \widehat{\mathbb{Z}_p^{ur}} = W(\mathbb{F}_p^a)$, $\delta x = \frac{\phi(x) - x^p}{p}$ (unique) $R^\delta = \{x \in R; x^{p^f} = x, f \geq 1\}$ $R = \{\sum_{i \geq 0} c_i p^i; c_i \in R^\delta\}$.
	Notation: $\widehat{\ }$ means p-adic completion
$f : R^N \to R$ is a δ-function of order $\leq n$ if there exists $P \in R[T, T', ..., T^{(n)}]$ ($T, T', ...$ N-tuples of variables) s.t. $f(u) = Pu := P(u, \delta u, ..., \delta^n u), u \in R^N$	$f : R^N \to \widehat{R}$ is a δ-function of order $\leq n$ if there exists $P \in R[T, T', ..., T^{(n)}]\widehat{\ }$ ($T, T', ...$ N-tuples of variables) s.t. $f(u) = Pu := P(u, \delta u, ..., \delta^n u), u \in R^N$

Globalizing: X smooth scheme over R $X(R) = \{R - \text{points of X}\}$ (If $R = C^\infty(N, \mathbb{C})$ X is the analogue of a submersion $M \to N$ of C^∞ manifolds and $X(R)$ is the analogues of the set $C^\infty_N(N, M)$ of sections of $M \to N$)	Globalizing: X smooth scheme over R $X(R) = \{R - \text{points of X}\}$
$... \to J^n(X) \to J^{n-1}(X) \to ... \to J^0(X) = X$ projective system of schemes (jet spaces)	$... \to J^n(X) \to J^{n-1}(X) \to ... \to J^0(X) = \widehat{X}$ projective system of p-adic formal schemes (p-jet spaces)
for X affine $X = Spec\ R[T]/(f)$, with T, f tuples $J^n(X) = Spec\ R[T, T', ..., T^{(n)}]/(f, \delta f, ..., \delta^n f)$ $B_n := R[T, T', ..., T^{(n)}]$, $B = \bigcup B_n$ $\delta : B \to B$ δ unique derivation lifting δ on R such that $\delta T = T'$, $\delta T' = T''$,...	for X affine $X = Spec\ R[T]/(f)$, with T, f tuples $J^n(X) = Spf\ R[T, T', ..., T^{(n)}]\hat{}/(f, \delta f, ..., \delta^n f)$ $B_n := R[T, T', ..., T^{(n)}]\hat{}$, $B = \bigcup B_n$ $\delta : B \to B$ unique p-derivation lifting δ on R such that $\delta T = T'$, $\delta T' = T''$,...
for X non-affine: glue	for X non-affine: glue
$\mathcal{O}(J^n(X)) \to Map(X(R), R)$ $[\Phi(x, x', ...)] \mapsto (\alpha \mapsto \Phi(\alpha, \delta\alpha, ...))$ $\mathcal{O}^n(X)$=image elements called δ-functions on X of order $\leq n$ (intuitively: algebraic diff equations)	$\mathcal{O}(J^n(X)) \to Map(X(R), \widehat{R})$ same $\mathcal{O}^n(X)$=image elements called δ-functions on X of order $\leq n$ (intuitively: arithmetic diff equations)

$\mathcal{O}^{n-1}(X) \subset \mathcal{O}^n(X)$ $\delta : \mathcal{O}^{n-1}(X) \to \mathcal{O}^n(X)$ total differential operator (Cartan distribution)	$\mathcal{O}^{n-1}(X) \subset \mathcal{O}^n(X)$ $\delta : \mathcal{O}^{n-1}(X) \to \mathcal{O}^n(X)$ arithmetic total differential operator
From now on, for convenience, R is a δ-closed field (i.e. *char* 0 and for any $f, g \in R[T, T', T'', ...]$ with T one variable and $ord(g) < ord(f)$ there exists $u \in R$ with $f(u, \delta u, ..) = 0$, $g(u, \delta u, ..) \neq 0$) Then $\mathcal{O}(J^n(X)) = \mathcal{O}^n(X)$	From now on $R = \widehat{\mathbb{Z}_p^{ur}}$ Then $\mathcal{O}(J^n(X)) = \mathcal{O}^n(X)$
Main problems 1) Compute $\mathcal{O}^n(X)$ for concrete X 2) Compute $\mathcal{O}^n(X)^\Gamma$ for a correspondence $\Gamma \to X \times X$ (classical differential invariant theory) gives algebro-geometric structure on X/Γ compare with A. Connes	Main problems same (without completion: uninteresting) same (analogue of differential invariant theory) same same
$J^1(X) \neq T(X)$, $J^n(X) \neq$arc spaces unless X descends to R^δ	$J^n(X) \otimes (R/pR)$=Greenberg transforms Note: δ does not descend to Greenberg transforms
$J^n(X) \to J^{n-1}(X)$ is a torsor for a vector bundle (hence a Zariski locally trivial fibration with fiber $\mathbb{A}^d$)	$J^n(X) \to J^{n-1}(X)$ is a Zariski locally trivial fibration with fiber an affine space $\widehat{\mathbb{A}}^d$.
$X = G$ group implies $J^n(G)$ groups $N^n := Ker(J^n(G) \to G)$ For G commutative $N^n \simeq \mathbb{G}_a^{nd}$ (as groups)	$X = G$ group implies $J^n(G)$ groups $N^n := Ker(J^n(G) \to \widehat{G})$ For G commutative $N^n \simeq \widehat{\mathbb{A}}^{nd}$ (as formal schemes) $N^n \not\simeq \widehat{\mathbb{G}}_a^{nd}$ as groups, in general. For $G = \mathbb{G}_a, \mathbb{G}_m, E$ (E elliptic curve) $N^1 \simeq \widehat{\mathbb{G}}_a$ (as groups) and there exists a homo $\chi : N^n \to \widehat{\mathbb{G}}_a^n$ $\chi = (\chi_1, ..., \chi_n)$, $\chi_1, ..., \chi_n$ linearly independent.

Theorem 1 (AB, AmerJM 93). $\mathcal{O}^n(\mathbb{P}^N) = R$.	**Theorem 1** (AB, Duke 96). $\mathcal{O}^n(\mathbb{P}^N) = R$.
Theorem 2 (Fuchs-Manin). For X elliptic curve there exists a δ-function $\phi : X(R) \to R$ of order ≤ 2 which is a non-zero homomorphism. If X defined over $R_0 \subset R$, $\delta R_0 \subset R_0$, $tr.deg.R_0/R_0^\delta = 1$ then $(Ker\ \psi) \cap X(R_0) = X(R_0)_{tors}$	**Theorem 2** (AB, Invent 95) For X elliptic curve there exists a δ-function $\phi : X(R) \to R$ of order ≤ 2 which is a non-zero homomorphism Moreover $[Ker\ \psi : p^\infty X(R)] < \infty$.
Proof. First part similar to $\to$ cf. AB $\neq$ Manin.	*Proof.* $0 \to N^2 \to J^2(X) \to \widehat{X} \to 0$ $Hom(J^2(X), \widehat{\mathbb{G}}_a) \to Hom(N^2, \widehat{\mathbb{G}}_a) \xrightarrow{\partial} H^1(X, \mathcal{O})$ there exist $a_1, a_2 \in R$, $\partial(a_1\chi_1 + a_2\chi_2) = 0$, etc.
similar statement for abelian schemes	similar statement for abelian schemes
Theorem 3 (Manin). X as in Theorem 2. There exists ψ of order 1 iff X descends to R^δ.	**Theorem 3** (AB, Invent 95). X as in Theorem 2. There exists ψ of order 1 iff X is a canonical lift (CL).
Theorem 4 (AB, AmerJM 94). X projective curve of genus ≥ 2 that does not descend to R^δ. Then $\mathcal{O}^1(X)$ separates the points of $X(R)$. In fact $J^n(X)$ are affine for $n \geq 1$.	**Theorem 4** (AB, Duke 96). X projective curve of genus ≥ 2. Then $\mathcal{O}^1(X)$ separates the points of $X(R)$. In fact $J^n(X)$ are affine for $n \geq 1$.
Theorem 4 embeds a projective curve into an affine space via δ-functions: $X(R) \to R^N$	Theorem 4 implies Manin-Mumford with effective bound, see below.

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Theorem 5 (AB, Duke 74).
$X/\mathbb{C}$ a curve of genus $g \geq 2$
in an abelian variety A, X not def $/\overline{\mathbb{Q}}$. Then
$\sharp(X \cap A_{tors}) \leq C(g)$

(finiteness conjectured by Manin-Mumford,
and proved by Raynaud
$C(g)$ conjectured by Mazur)

Proof Heavily uses δ.Skipped.

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Theorem 5 (AB, Duke 96).
$X/\mathbb{C}$ a curve of genus $g \geq 2$
in its Jacobian A, X def $/\overline{\mathbb{Q}}$. Then
$\sharp(X \cap A_{tors}) \leq C(g,p)$
where p smallest prime of good reduction

(finiteness conjectured by Manin-Mumford
and proved by Raynaud
$C(g)$ conjectured by Mazur)

Proof. May replace $\mathbb{C}$ by R
Enough $\sharp(X(R) \cap A(R)_{prime-to-p-tors})$
$\leq C(g,p)$
Any point P in this set lifts to
an R-point $J^1(P)$ of $J^1(X) \subset J^1(A)$
The reduction mod p $\overline{J^1(P)}$ of $J^1(P)$ lies in
$\overline{J^1(X)} \cap p\overline{J^1(A)}$.
This $\cap$ is finite because it's affine $\cap$ projective
Cardinality bounded by Bezout.

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Theorem 6 (AB, AmerJM 95).
X as in Theorem 4. Then the δ-functions

$X(R) \subset Jac(X)(R) \xrightarrow{\psi} R$

generate the field $Frac(\mathcal{O}^1(X))$ over R
(but not necessarily the R-algebra $\mathcal{O}^1(X)$.)

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3. ODEs, Part II

δ-modular forms: theory less rich but still interesting (cf. eg. Ramanujan, AB, Crelle 2000.)	arithmetic δ-modular forms: rich theory: cf. below. (AB, Crelle 2000, Compositio 2003, 2004, 2009.)
	$X \subset X_1(N)$ open, disjoint from cusps $X(R) \subset \{(A, \alpha); A/R \text{ an elliptic curve}, \alpha \text{ a level } \Gamma_1(N) \text{ structure}\}$ $\mathcal{E} \xrightarrow{\pi} X$ universal elliptic curve $L := \pi_* \Omega_{\mathcal{E}/X}$ $V := Spec\left(\bigoplus_{n \geq 0} L^{\otimes n}\right)$ $V^* := V \backslash O$ a $\mathbb{G}_m$-torsor $V^*(R) \subset \{(A, \alpha, \omega); \omega \text{ a 1-form on } A\}$
	δ-modular function of order $\leq n$: any element of $M^n := \mathcal{O}^n(V^*)$ viewed as a map $V^*(R) \to R$ $M^\infty := \bigcup_n M^n$
	$W := \mathbb{Z}[\phi] = \{\sum a_i \phi^i; a_i \in \mathbb{Z}\}$ $deg : W \to \mathbb{Z}$, $deg(\sum a_i \phi^i) = \sum a_i$ δ-modular form of weight $w \in W$: any $f \in M^n$, $f : V^*(R) \to R$ such that $f(\lambda \cdot a) = \lambda^w f(a)$, $\lambda \in R^\times, a \in V^*(R)$ $M^n(w) = \{\delta\text{-modular forms of weight } w \text{ and order} \leq n\ \}$.

	Let $w \in W$ with $deg(w)$ even. $f \in M^n(w)$ is isogeny covariant if for any $(A_1, \alpha_1, \omega_1), (A_2, \alpha_2, \omega_2)$ representing $P_1, P_2 \in V^*(R)$ and any isogeny $u : A_1 \to A_2$ of degree prime to p with $u^* \omega_2 = \omega_1$ we have $f(P_1) = (deg(u))^{-deg(w)/2} f(P_2)$. $I^n(w) := \{f \in M^n(w); f \text{ isogeny covariant}\}$, $I(w) = I^{ord(w)}(w)$ NB. $Proj\left(\bigoplus_w I(w)\right)$ is "morally" $\frac{X_1(N)}{\text{Hecke correspondences}}$
	Aim of the theory: to compute $M^n(w), I^n(w), M^{\infty}, ...$
	Fourier (expansion) map $M^{\infty} \to \widehat{R((q))}$
analogue of f^1 (related to Ramanujan) but no analogue of f^{∂}	**Theorem 7** (AB+Barcau+Saha). Assume X is in the ordinary locus. Then $\bigoplus_w I(w)$ δ-generated by 2 forms $f^1 \in M^1(-1-\phi)$, $f^{\partial} \in M^1(\phi - 1)$ Moreover: 1) The kernel of the Fourier map is δ-generated by f^1 and $f^{\partial} - 1$ 2) The p-adic closure of the image of the Fourier map is Katz's ring $\mathbb{W}$ of generalized p-adic modular forms.
	moral: the divided congruences of Katz can all be obtained from arithmetic differential objects.

$y^2 = x^3 + a_4 x + a_6$, a_4, a_6 indeterminates	same same
$\Delta = \Delta(a_4, a_6)$ the discriminant	same same
$\mathbb{C}[a_4, a_6, \Delta^{-1}] \to \mathcal{O}(V^*)$	$R[a_4, a_6, \Delta^{-1}] \to \mathcal{O}(V^*)$
$\mathbb{C}[a_4, a_6, a_4', a_6', \Delta^{-1}] \to \mathcal{O}^1(V^*)$	$R[a_4, a_6, a_4', a_6', \Delta^{-1}]\hat{} \to \mathcal{O}^1(V^*)$
$a_4' \mapsto q\frac{d}{dq}(a_4(q)) \in \mathbb{C}((q))$ via Fourier	$a_4' \mapsto \frac{a_4(q^p) - a_4(q)^p}{p} \in \widehat{R((q))}$ via Fourier

$f^1 = \frac{2a_4a_6' - 3a_6a_4'}{\Delta}$ (easy) $f^1 \mapsto 1$ via Fourier by Ramanujan	$f^1 \equiv E_{p-1}\frac{2a_4^p a_6' - 3a_6^p a_4'}{\Delta^p} + f_0(a_4, a_6) \bmod p$ cf. Hurlburt, Compositio 2001 $f^1 \mapsto 0$ via Fourier f_0 related to Kronecker's modular polynomial mod p^2
Jet construction of f^1 (via usual Kodaira-Spencer)	Jet construction of f^1 universal elliptic curve $E = \bigcup U_i \to Spec\ M^\infty$ $s_i : \widehat{U}_i \to J^1(U_i)$ sections of the natural projection $s_i - s_j : \widehat{U}_i \cap \widehat{U}_j \to N^1 \simeq \widehat{\mathbb{G}}_a$ get class $\eta \in H^1(\widehat{E}, \mathcal{O}) = H^1(E, \mathcal{O})$ $f^1 =$ Serre dual of $\eta \cup \frac{dx}{y}$.
	Note $f^1(P) = 0$ iff P has CL
	Construction of f^∂ (Barcau, Compositio 2002) $f^\partial = \text{const} \cdot (72(a_6^p + pa_6')\frac{\partial}{\partial a_4'}$ $-16(a_4^p + pa_4')^2\frac{\partial}{\partial a_6'} - p(P^p + p\delta P))(f^1)$ P the Ramanujan form viewed as p-adic modular form, $P \in R[a_4, a_6, \Delta^{-1}, E_{p-1}^{-1}]\widehat{\ }$
	both f^1 and f^∂ admit crystalline constructions
	Theorem 8 (AB+Poonen, Duke 2009) S modular curve, A elliptic curve, X curve over $\mathbb{C}$, $\pi : X \to S$, $\varphi : X \to A$. $CM \subset S$ CM locus, $\Gamma \subset A$ subgroup $rank(\Gamma) := dim_{\mathbb{Q}}(\Gamma \otimes \mathbb{Q}) < \infty$. Then $\sharp(\pi^{-1}(CM) \cap \varphi^{-1}(\Gamma)) < \infty$.

No interesting analogue	**Theorem 9** (AB+Poonen, Compositio 2009) S, A, X, π, φ as above and over $R = \widehat{\mathbb{Z}_p^{ur}}$ $CL \subset S(R)$ CL locus. Then there exists a constant c such that for any subgroup $\Gamma \subset A(R)$ $\sharp(\pi^{-1}(CL) \cap \varphi^{-1}(\Gamma)) < c \cdot p^{rank(\Gamma)}$ *Proof.* (Case $rank(\Gamma) = 0$, $\varphi = id$) AB, Invent 95 gives homo $\psi : A(R) \to R$ of order 2 Let $f^\sharp = \psi \circ \varphi : S(R) \to R$ AB, Crelle 2000 gives $f^\flat : S^\dagger(R) \to R$ (constructed from f^1 above, $S^\dagger \subset S$) vanishing on CL. So any P in intersection is a solution of the system $f^\sharp = f^\flat = 0$ Claim: there are $h_0, h_1 : S^\dagger(R) \to R$ such that $f^\sharp - h_0 f^\flat - h_1 \delta f^\flat$ has order 0. (I.e. one can eliminate the derivatives in the system of differential equations to get an equation without derivatives.) The latter has only finitely many zeros (Strassman) and P is one of them.

4. PDEs: hyperbolic and parabolic type

Fix $R = C^\infty(\mathbb{R}_x, \mathbb{C})$ with coordinate x Fix $A = C^\infty(\mathbb{R}_x \times \mathbb{R}_t, \mathbb{C})$ with coordinates x, t $(A, \partial_x, \partial_t)$ $\partial_x u = \frac{\partial u}{\partial x}$ $\partial_t u = \frac{\partial u}{\partial t}$	Fix $R = \widehat{\mathbb{Z}_p^{ur}}$ with "coordinate" p $A = R[[q]]$ with "coordinates" p, q (A, δ_p, δ_q) $\delta_p u = \frac{u^{(\phi)}(q^p) - u(q)^p}{p}$ $\delta_q u = q\frac{\partial u}{\partial q}$
$P : A \to A$ $Pu = P(t, x, u, ..., \partial_x^i \partial_t^j u, ...)$ P polynomial in u and the partials more generally same for manifolds	$P : A \to A$ $Pu = P(u, ..., \delta_p^i \delta_q^j u, ...)$ P a p-adic limit of polynomials more generally $P : X(A) \to A$ for X smooth scheme over A.
$P = \psi$ called linear with constant coefficients if $\psi u = \sum c_{ij} \partial_x^i \partial_t^j u$ $c_{ij} \in \mathbb{C}$ symbol: $\sigma_P = \sum c_{ij} \sqrt{-1} \xi^i \tau^j$ $\psi = \psi^r$ means ψ has order r $\psi = \psi_x$ means ψ involves ∂_x only same with t $\psi = \psi_{xt}$ means ψ involves both ∂_x and ∂_t	For $X = G$ group $P = \psi$ called linear if $\psi : G(A) \to A$ a homomorphism (and same for any extension of A on which δs operate with appropriate comm rel) there is an analogue $\psi = \psi^r$ means ψ has order r $\psi = \psi_p$ means ψ involves δ_p only same with q $\psi = \psi_{pq}$ means ψ involves both δ_p and δ_q
	Concerned with $G = \mathbb{G}_a, \mathbb{G}_m, E$ sometimes write ψ_a, ψ_m, ψ_E

Examples of P/classification $\partial_t u - \partial_x u$ (convection) $\partial_t u - \partial_x^2 u$ (heat) $\partial_t^2 u - \partial_x^2 u$ (wave) $\partial_t^2 u + \partial_x^2 u$ (Laplace, no arith analogue)	**Theorem 10** (AB+Simanca) 1) For $\mathbb{G}_a$ all ψs are built from Id via ϕ_p and δ_q 2) For $\mathbb{G}_m$ all ψs built from $\psi_p^1 u = \frac{\delta_p u}{u^p} - \frac{p}{2}\left(\frac{\delta_p u}{u^p}\right)^2 + ...$ $\psi_q^1 u = \frac{\delta_q u}{u}$ 3) For E/A general: a) No analogues of ψ_p^1, ψ_q^1 b) SURPRISE !!! There is a ψ_{pq}^1 (convection eq) c) There is a ψ_q^2 (Manin 63) and a ψ_p^2 (AB, Invent 95) d) All ψs are built from the above e) one relation $\psi_q^2 + \lambda\psi_p^2 = \psi_a^1 \circ \psi_{pq}^1$ (canonical wave eq; λ unique, interesting invariant) 4) For E/R general $/R$ all ψs built from ψ_q^1, ψ_p^2 (in particular heat eqs $\psi_q^1 + \lambda\psi_p^2$, λ variable.)
solution space $\mathcal{U} = \{u \in A; \psi u = 0\}$	solution space $\mathcal{U} = \{u \in G(A); \psi u = 0\}$
stationary solutions: $\partial_t u = 0$	stationary solutions (cases 1,2,4 above) $\psi_q u = 0$ (i.e. $u \in G(R)$) (there is also a good definition for case 3)
no analogue, problem trivial	**Theorem 11** (AB+Simanca) Complete classification of ψs that admit non-stationary solutions (Morally a quantization phenomenon: e.g. heat equation in 4 above has non-stationary solutions iff λ is in the set of all $\mathbb{Z}$-multiples of a certain element in R.)
Convolution. the "ring" $R_\star$ is the group R with "multiplication" $(f \star g)(x) = \int f(y)g(x-y)dy$ $A_\star$ is the group A viewed as $R_\star$-"module" under $\star$ $\mathcal{U}$ is a $R_\star$-"submodule" " " because convolution not always defined; also need distributions....	*Convolution* $R_\star := \mathbb{Z}\mu(R) = \{f : \mu(R) \to \mathbb{Z}; \text{finite support}\}$ where $\mu(R)$ group of roots of unity in R $R_\star$ group ring under convolution $R_\star \to R$, $f \mapsto \sum f(\zeta)\zeta$, ring homo $G(A)$ is a $R_\star$-module under $\star$: $f \star u = \sum_\zeta f(\zeta)\sigma_\zeta u$ $\sigma_\zeta : G(A) \to G(A)$ induced by $\sigma_\zeta : A \to A$, $q \mapsto \zeta q$ $\mathcal{U}$ is a $R_\star$-submodule (no reason a priori to be an R-module)

Fundamental solutions For $P = \psi$ with constant coefficients If $\mathcal{U} \subset A \to R^\rho$ $u \mapsto (u, \partial_t u, ...,)\|_{t=0}$ is a bijection (boundary value well posed) then $\mathcal{U}$ is a free $R_\star$-"module" under $\star$ of rank ρ (basis of fundamental solutions is a basis mapped to Dirac $\times$ identity matrix)	**Theorem 12** (AB+Simanca) Let $\psi : G(A) \to A$ be non-degenerate (a condition on the symbol $\sigma(\xi, tau)$) Let $\mathcal{U}_1$ be the group of solutions vanishing at $q = 0$ Then convolution module structure of $\mathcal{U}_1$ descends to an R-module structure and $\mathcal{U}_1$ is a finitely generated free R-module Its rank is the number of positive integer roots of $\sigma(0, \tau)$
Exponential solutions $Pu = 0$ gives by Fourier inversion in x $\sigma_P(-\xi, -\sqrt{-1}\partial_t)\widehat{u}(\xi, t) = 0$ ordinary diff eqn with parameter ξ its solutions are linear combinations of exponentials again by Fourier inversion $u(x,t) = \sum_{j=1}^{\rho} \int C_j(\xi) e^{-\sqrt{-1}\xi x - \sqrt{-1}\tau_j(\xi) t} d\xi$ $\tau_j(\xi)$ roots of $\sigma_P(-\xi, -\tau)$.	Above can be also viewed as an analogue of exponential solutions

5. PDEs: elliptic type

$A = C^{\infty}(D, \mathbb{C})$, $D \subset \mathbb{C}$, $z, \overline{z} \in A$ $\partial_z, \partial_{\overline{z}} : A \to A$	$A = \mathbb{Z}, \mathbb{Z}[\zeta_m, 1/N]$, $p_1, p_2 \in A$ $\delta_{p_1}, \delta_{p_2} : A \to A$ $\delta_{p_i} a = \frac{\phi_{p_i}(a) - a^{p_i}}{p_i}$
$M \to D$ a C^{∞} submersion $M(A) := C^{\infty}_D(D, M)$ set of sections	$X \to Spec\ A$ smooth scheme $X(A)$ set of sections i.e. of A-points
$\psi : M(A) \to R$ (non linear diff operators) $u \mapsto \psi u = P(z, \overline{z}, u, ..., \partial_z^i \partial_{\overline{z}}^j u, ...)$ P a polynomial in u and the partials B the ring of all such operators If ψ has order r write $\psi = \psi^r$ If ψ only involves z write $\psi = \psi_z$	$\psi : X(A) \to R$ (non linear diff operators) $u \mapsto \psi u = P(u, ..., \delta_{p_1}^i \delta_{p_2}^j u, ...)$ P a polynomial B the ring of all such operators (need to allow variable A) If ψ has order r write $\psi = \psi^r$ If ψ only involves p_1 write ψ_{p_1}
If $M = G$ family of Lie groups over D and ψ homo, ψ called linear	If $X = G$ is a group scheme over A and ψ homo we COULD call ψ a linear arithmetic differential operator PROBLEM: B does not contain such non-zero ψs in most cases SO WE NEED ANOTHER DEFINITION
	FACT: In many cases for any affine $X = Spec\ B \subset G$ the completions $B^{\widehat{p_1}}$ and $B^{\widehat{p_2}}$ in the p_1-adic and p_2-adic topologies contain non-zero "linear elements" ψ_{p_1} and ψ_{p_2} respectively. AIM: to "analytically continue" ψ_{p_1} and ψ_{p_2} DILLEMA; ψ_{p_1} and ψ_{p_2} are defined on disjoint spaces $X^{\widehat{p_1}}$ and $X^{\widehat{p_2}}$

No problem here	Main idea: assume for simplicity $A = \mathbb{Z}[1/N]$ and set $A_0 = \mathbb{Z}_{(p_1)} \cap \mathbb{Z}_{(p_2)}$, $B_0 = B \otimes_A A_0$ say ψ_{p_1} and ψ_{p_2} can be analytically continued along a section $P \in X(A)$ (with ideal I in B) if there exists $\psi_0 \in B_0^{\widehat{I}}$ which coincides with ψ_{p_1} and ψ_{p_2} in the rings $B_0^{\widehat{(I,p_1)}}$ and $B_0^{\widehat{(I,p_2)}}$ respectively. (PICTURE!!!!)
	one may assume for all practical purposes that $B_0^{\widehat{I}} = A_0[[t]]$, t a tuple and what we require is that there is an element in $A_0[[t]]$ whose images in $\mathbb{Z}_{p_1}[[t]]$ and $\mathbb{Z}_{p_2}[[t]]$ coincide with the images of ψ_{p_1} and ψ_{p_2}
	DEFINITION: a linear arithmetic partial differential operator is a pair $\psi = (\psi_{p_1}, \psi_{p_2})$ of linear elements as above that can be analytically continued along the zero section of G (Write $\psi : G \to \mathbb{G}_a$.)
Example $G = \mathbb{C}^\times \times D \to D$ $G(R) = C^\infty(D, \mathbb{C}^\times)$ $\mathbb{G}_a(R) = R = C^\infty(D, \mathbb{C})$ $\psi^2 : C^\infty(D, \mathbb{C}^\times) \to C^\infty(D, \mathbb{C})$ $\psi^2 u = \frac{1}{4} \Delta \log u = \partial_z \partial_{\overline{z}} u$	*Example* $\mathbb{G}_m \to Spec\ A$ $\psi : \mathbb{G}_m \to \mathbb{G}_a$ $\psi^2_{p_1} u = \left(1 - \frac{\phi_{p_2}}{p_2}\right) \sum (-1)^{n+1} \frac{p_1^n}{n} \left(\frac{\delta_{p_1} u}{u^{p_1}}\right)^n$ $\psi^2_{p_2} = \left(1 - \frac{\phi_{p_1}}{p_1}\right) \sum (-1)^{n+1} \frac{p_2^n}{n} \left(\frac{\delta_{p_2} u}{u^{p_2}}\right)^n$ $\psi^2_{p_1} \in \mathbb{Z}_{p_1}[x, x^{-1}, \delta_{p_1} x, \delta_{p_2} x, \delta_{p_1}\delta_{p_2} x]^{\widehat{p_1}}$ $\psi^2_{p_2} \in \mathbb{Z}_{p_2}[x, x^{-1}, \delta_{p_1} x, \delta_{p_2} x, \delta_{p_1}\delta_{p_2} x]^{\widehat{p_2}}$ They can be analytically continued because they come from the same series in $A_0[[T, \delta_{p_1} T, \delta_{p_2} T, \delta_{p_1}\delta_{p_2} T]]$ via $x \mapsto T + 1$

$\psi^2 u = \partial_z \left(\frac{\partial_{\overline{z}} u}{u}\right) = \partial_{\overline{z}} \left(\frac{\partial_z u}{u}\right)$ ("Dirac decomposition")	the analytic continuation above is an analogue of the "Dirac decomposition"
	Theorem 13 (AB+Simanca, Advances Math 2009). All linear arithmetic partial differential operators on $\mathbb{G}_m$ are obtained from ψ^2 above.
Example universal elliptic curve over D $E = (D \times \mathbb{C})/\sim \to D$ $\psi^4 : C^\infty_D(D, E) \to C^\infty(D, \mathbb{C})$ $\psi^4 u = \frac{1}{16} \Delta\Delta \log_E u = \partial_z^2 \partial_{\overline{z}}^2 \log_E u$ where $\log_E : E - -- > D \times \mathbb{C} \to \mathbb{C}$ is the multivalued logarithm (again, Dirac decomposition)	*Example* elliptic curve $E \to Spec\ A$ $\psi^4 : E \to \mathbb{G}_a$ $\psi^4_{p_1} = \left(1 - a_{p_2} \frac{\phi_{p_2}}{p_2} + p_2 \left(\frac{\phi_{p_2}}{p_2}\right)^2\right) \psi^2_{p_1}$ $\psi^4_{p_2} = \left(1 - a_{p_1} \frac{\phi_{p_1}}{p_1} + p_1 \left(\frac{\phi_{p_1}}{p_1}\right)^2\right) \psi^2_{p_2}$ (L factors); again these can be analytically continued along the origin which is an analogue of Dirac decomposition.
	Theorem 14 (AB+Simanca, Advances Math 2009). If E has ordinary reduction at p_1, p_2 then all linear arithmetic partial differential operators on E are obtained from ψ^4 above.
	Another view on analytic continuation between primes: cf. Borger+AB