Effective potential and gauge invariance in FLRW

with Damien George and Marieke Postma

Sander Mooij
Nikhef, Amsterdam
Supervisor: Marieke Postma

July 20, 2012
Lorentz Center workshop, Leiden
Introduction

• “Precision cosmology era” calls for precise understanding of scalar field dynamics

• Birrell & Davies ’78–’79 ... one-loop effective action for a scalar field in FLRW

• Here: scalar field couples to U(1) gauge field
Goal of project

• Compute (unrenormalized) divergent contributions to the one-loop effective action of a U(1) charged scalar (Abelian Higgs model)

• Allow for time-dependent masses

• Expanding FLRW background

• Show on-shell gauge invariance
Relevance

- models with SM/GUT Higgs as inflaton/waterfall field

- flat directions in MSSM
\( \mathcal{L} = \mathcal{L}_{\text{gaugekinetic}} + \mathcal{L}_{\text{higgs kinetic}} + \mathcal{L}_{\text{pot}} + \mathcal{L}_{\text{gaugefixing}} + \mathcal{L}_{\text{faddeev–popov}} \)

\[
\begin{align*}
\mathcal{L} & = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi (D^\mu \Phi)^\dagger - V(\Phi \Phi^\dagger) - \frac{1}{2\xi} G^2 + \bar{\eta} g \frac{\delta G}{\delta \alpha} \eta \\
\left( G = \partial_\mu A^\mu - \xi g (\phi + h) \theta \right) \\
\Phi(t, \vec{x}) & = \frac{1}{\sqrt{2}} \left[ \phi(t) + h(t, \vec{x}) + i \theta(t, \vec{x}) \right]
\end{align*}
\]
Effective action

- Quantum corrected classical action

- Quantum corrections come from vacuum loops:

\[ L = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) - V_{\text{CW}}(\phi) \]

- Integrate over all fluctuating fields, interested in effect on background

- Masses in propagators or as interactions
Equation of motion

- “Tadpole method” (Weinberg ’74)

\[
0 = \partial_{\mu} \partial^{\mu} \phi + \frac{\partial V}{\partial \phi} + \frac{\partial V_{CW}}{\partial \phi}
\]

- In the end, integrate back w.r.t. to \( \phi \) to get effective potential
Three cases

• static background (standard Coleman-Weinberg)

• time dependent background

• FLRW
Static background

• Original CW set-up: fixed background value, \( \varphi(t)=\text{constant} \)

\[
\mathcal{L}_{\text{cl}} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi)
\]

\[
\mathcal{L}_{\text{free}} = -\frac{1}{2} A^\mu \left[ -\eta_{\mu\nu} (\partial^2 + g^2 \varphi^2) + \partial_\mu \partial_\nu (1 - \frac{1}{\xi}) \right] A^\nu - \bar{\eta} \left[ \partial^2 + \xi g^2 \varphi^2 \right] \eta
\]

\[
-\frac{1}{2} h \left[ \partial^2 + V_{hh} \right] h - \frac{1}{2} \theta \left[ \partial^2 + V_{\theta\theta} + \xi g^2 \varphi^2 \right] \theta
\]

\[
\mathcal{L}_{\text{int}} = -h \left[ \partial^2 \varphi + V_\varphi \right] + \ldots
\]
Example: Higgs loop

- Can put all masses in propagator

- No two-point interactions left

\[ h \quad \lambda_h \]

\[ \partial_\mu \partial^\mu \phi + \frac{\partial V}{\partial \phi} \]

\[ h \quad \lambda_{hhh} \quad D_h \]

\[ -i \int d^4 x \frac{1}{2} \partial_\phi (-i m^2_h) D_h(x - x) \]

Together:

\[ 0 = \partial_\mu \partial^\mu \phi + \frac{\partial V}{\partial \phi} + \frac{1}{16\pi^2} \partial_\phi m^2_h \left( \Lambda^2 - m^2_h \ln \left( \frac{\Lambda}{m_h} \right) \right) \]
Total time-independent result

\[ V_{CW} = \frac{\Lambda^2}{16\pi^2} \left( V_{hh} - 2\xi g^2 \phi_{cl}^2 + V_{\theta\theta} + \xi g^2 \phi_{cl}^2 + 3g^2 \phi_{cl}^2 + \xi g^2 \phi_{cl}^2 \right) \]

\[ - \frac{\ln(\Lambda/m)}{32\pi^2} \left( V_{hh}^2 - 2\xi^2 g^4 \phi_{cl}^4 + V_{\theta\theta}^2 + \xi^2 g^4 \phi_{cl}^4 + 2V_{\theta\theta} \xi g^2 \phi_{cl}^2 + 3g^4 \phi_{cl}^4 + \xi^2 g^4 \phi_{cl}^4 \right) \]

\[ = \frac{\Lambda^2}{16\pi^2} \left( V_{hh} + V_{\theta\theta} + 3g^2 \phi_{cl}^2 \right) \]

\[ - \frac{\ln(\Lambda/m)}{32\pi^2} \left( V_{hh}^2 + V_{\theta\theta}^2 + 3g^4 \phi_{cl}^4 + 2V_{\theta\theta} \xi g^2 \phi_{cl}^2 \right) \]

The final result is gauge dependent.
Going on-shell

- Goldstone (potential is function of $|\Phi|^2$)

\[
\left. \frac{\partial^2 V}{\partial \theta^2} \right|_{\text{cl}} = \frac{1}{\phi} \left. \frac{\partial V}{\partial \phi} \right|_{\text{cl}}
\]

- Klein-Gordon equation of motion:

\[
\ddot{\phi} + \frac{\partial V}{\partial \phi} = 0
\]

- Therefore:

\[
\phi = \text{const} \quad \Rightarrow \quad V_{\theta\theta} = 0
\]

- also: Nielsen identity

\[
\left( \text{Nielsen 1975} \right) \quad \frac{\partial V_{\text{eff}}}{\partial \xi} + \frac{\partial \phi}{\partial \xi} \frac{\partial V_{\text{eff}}}{\partial \phi} = 0
\]
Final Coleman-Weinberg result

\[ V_{CW} = \frac{\Lambda^2}{16\pi^2} \left( V_{hh} + 3g^2 \phi_{cl}^2 \right) - \frac{\ln(\Lambda/m)}{32\pi^2} \left( V_{hh} + 3g^4 \phi_{cl}^4 \right) \]
Three cases

- static background (standard Coleman-Weinberg)
- time dependent background
- FLRW
Time-dependent masses \((1104.4897)\)

- For example: Higgs inflation

\[ m^2(t) = \bar{m}^2 + \delta m^2(t) \]
\[ \delta m^2(0) = 0 \]

- Consider effective action instead of potential

- Split masses in time-independent background and time-dependent interaction

- In-in formalism
\[ \mathcal{L}_{\text{cl}} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) \]

\[ \mathcal{L}_{\text{free}} = -\frac{1}{2} A^\mu \left[ -g_{\mu\nu} (\partial^2 + g^2 \phi_0^2) + \partial_{\mu} \partial_{\nu} (1 - \frac{1}{\xi}) \right] A^\nu - \bar{\eta} \left[ \partial^2 + \xi g^2 \phi_0^2 \right] \eta \]

\[ -\frac{1}{2} h [\partial^2 + V_{hh}(0)] h - \frac{1}{2} \theta \left[ \partial^2 + V_{\theta\theta}(0) + \xi g^2 \phi_0^2 \right] \theta \]

\[ \mathcal{L}_{\text{int}} = -h [\partial^2 \phi + V_{\phi}] + \frac{g^2}{2} (\phi(t)^2 - \phi_0^2) [A_\mu A^\mu - \xi \theta^2 - 2\xi \bar{\eta} \eta] \]

\[ -2g \partial_{\mu} \phi(t) A^\mu \theta - \frac{1}{2} (V_{hh}(t) - V_{hh}(0)) h^2 - \frac{1}{2} (V_{\theta\theta}(t) - V_{\theta\theta}(0)) \theta^2 + \ldots, \]

new coupling
Higgs loop

\[ \partial_{\mu} \partial^{\mu} \phi + \frac{\partial V}{\partial \phi} \]

\[ -i \int d^4 x \frac{1}{2} \partial \phi \left( -i \delta m_h^2 \right) \bigg|_{x^0} D_{h^+}^{++}(x - x) \]

\[ -i \int d^4 x \int d^4 y \frac{1}{2} \partial \phi \left( -i \delta m_h^2 \right) \bigg|_{x^0} \left( D_{h^+}^{++}(x - y) \right) \bigg|_{y^0} \left( D_{h^+}^{--}(x - y) \right) \]

integrate by parts, use boundary condition

together:

\[ 0 = \partial_{\mu} \partial^{\mu} \phi + \frac{\partial V}{\partial \phi} + \frac{1}{16\pi^2} \partial \phi m_h^2(t) \left( \Lambda^2 - m_h^2(t) \ln \left( \Lambda / \bar{m}_h \right) \right) \]
Mixed Goldstone-gauge boson loop

\[-i \int d^4 x \int d^4 y \partial \phi \left( -i \delta m^2_{A\theta} \right) \bigg|_{x^0} \times
\]

\[
\left( D_{++ A\nu}^{+} (x-y) \eta^{0\mu} \eta^{0\nu} \left( -i \delta m^2_{A\theta} \right) \bigg|_{y^0} D_{0}^{++} (y-x) + D_{-- A\nu}^{+} (x-y) \eta^{0\mu} \eta^{0\nu} \left( i \delta m^2_{A\theta} \right) \bigg|_{y^0} D_{0}^{-+} (y-x) \right)
\]

= ... \\
= ... \\
= \frac{3 + \xi}{32 \pi^2} (\partial_\phi \delta m^2_{A\theta}) \delta m^2_{A\theta} \ln (\Lambda/\bar{m})

Figure 1: Tree level tadpole giving classical equation of motion and the radiative order. The summation is over all fields and also over Lorentz indices and over the gauge bosons. The 1-loop equation of motion follows from the vanishing of the tadpole, see Appendix ?? for details. These are all one particle irreducible tadpole graphs with one external leg. We compute these diagrams and thus the quantum corrected equation of motion at the loop level. Only diagrams with up to three vertices contribute to the UV divergent terms, on which we concentrate. The calculation is done in the conformal frame, in terms of hatted fields and mass scales. For notational convenience, in this section we drop the hat on all quantities; it shall be reinstated at the end to give the results. The calculation is analogous to the one for Minkowski, but with mass terms that now depend on the FRW scale factor. This is straightforward to incorporate for the diagrams with a scalar running in the loop. There are however some new technical difficulties that come in with the gauge boson loops: the mass of the temporal gauge boson gets FRW corrections but the mass of the spatial components does not. This is possible because Lorentz symmetry is broken by the time-dependent background. Consequently the diagram with $A_0$ and $A_i$ contribute differently. The off-diagonal gauge boson mass is non-zero. This results in new diagrams with both $u$ and $v$ mass insertions. The formalism set up such that the uppoint interactions vanish at the initial time. This avoids divergencies which depend on the initial conditions. We will argue in section ?? that this is always decay for arbitrary initial conditions, provided the initial vacuum is chosen accordingly. The oneloop equations of motion can be extracted from the series of tadpole diagrams (suppressing +- indices).
Taking all diagrams together

\[ \Gamma = \int d^4 x \left[ \frac{1}{2} \partial_\mu \phi_{cl} \partial^\mu \phi_{cl} - V(\phi_{cl}) \right. \]

\[ \left. - \frac{\Lambda^2}{16\pi^2} \left( V_{hh}(t) - 2\xi g^2 \phi_{cl}(t)^2 + V_{\theta\theta}(t) + \xi g^2 \phi_{cl}(t)^2 + 3g^2 \phi_{cl}(t)^2 + \xi g^2 \phi_{cl}(t)^2 \right) \right. \]

\[ \left. + \frac{\ln (\Lambda/\bar{m})}{32\pi^2} \left( V_{hh}^2(t) - 2\xi^2 g^2 \phi_{cl}(t)^4 + V_{\theta\theta}^2(t) + \xi^2 g^2 \phi_{cl}(t)^4 + 2V_{\theta\theta}(t)\xi g^2 \phi_{cl}(t)^2 \right. \right. \]

\[ \left. + 2V_{\theta\theta}(t)\xi g^2 \phi_{cl}(t)^2 \right) \left[ 3g^4 \phi_{cl}(t)^4 + \xi g^4 \phi_{cl}(t)^4 - (6 + 2\xi) g^2 \phi_{cl}(t)^2 \right] \]
Again: going on-shell

- Equation of motion:

\[ g^2 \int dt \dot{\phi}^2 = -g^2 \int dt \phi \ddot{\phi} = g^2 \int dt \phi V_{\phi} \]

- Goldstone

\[ g^2 \int dt \phi V_{\phi} = g^2 \int dt \phi^2 V_{\theta\theta} \]

woensdag 25 juli 2012
Gauge-invariant final result

\[ \Gamma = \int d^4x \left[ \frac{1}{2} \partial_\mu \phi_{cl} \partial^\mu \phi_{cl} - V(\phi_{cl}) \right. \]

\[ \left. - \frac{\Lambda^2}{16\pi^2} \left( V_{hh}(t) + V_{\theta\theta}(t) + 3g^2 \phi_{cl}(t)^2 \right) \right. \]

\[ \left. + \frac{\ln(\Lambda/\bar{m})}{32\pi^2} \left( V_{hh}^2(t) + V_{\theta\theta}^2(t) + 3g^4 \phi_{cl}(t)^4 - 6g^2 \phi_{cl}(t)^2 V_{\theta\theta}(t) \right) \right] \]
Unitary gauge?

• Final expression contains nonzero $V_{\theta \theta}$

• However, all reference to $\theta$ can still be removed in unitary gauge

• Problem is in use of unitary gauge: have to add extra term

  Grosse-Knetter & Kogerler '92

• SUSY Higgs inflation

  Linde et al. 2010
Three cases

• static background (standard Coleman-Weinberg)

• time dependent background

• FLRW
Extension to FLRW

\[ ds^2 = a^2(\tau) \left[ d\tau^2 - d\vec{x}^2 \right] \]

- conformal scaling of fields and masses
  \[ \hat{\psi}_i = a\psi_i, \quad \psi_i = \{\phi, h, \theta, \eta\} \]
  \[ \hat{m} = am \]
  \[ \hat{V} = a^4V \]

- new interactions

- need to go to third order
FLRW: quadratic action

\[ S^{(2)} = \int d^4x \left\{ -\frac{1}{2} A_\mu \left[ -(\partial^2 + \hat{m}_{(\mu)}^2) \eta^{\mu\nu} + \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right] A_\nu - A_0 (\hat{m}^2)^{i0} A_i \right\} \]

\[ -\hat{m}_{A_\theta}^2 A_0 \hat{\theta} - \frac{1}{2} \sum_{\phi_i = \{\phi, \hat{\theta}\}} \hat{\phi}_i \left( \partial^2 + \hat{m}_{\phi_i}^2 \right) \hat{\phi}_i - \hat{\eta} \left( \partial^2 + \hat{m}_{\eta}^2 \right) \hat{\eta} \right\} \]

\[ \hat{m}_{(\mu)}^2 = g^2 \hat{\phi}^2 + \frac{2}{\xi} (H' - 2H^2) \delta_{\mu0} \]

\[ \hat{m}_{h}^2 = \hat{V}_{hh} - (H' + H^2) \]

\[ \hat{m}_{\theta}^2 = \hat{V}_{\theta\theta} + \xi g^2 \hat{\phi}^2 - (H' + H^2) \]

\[ \hat{m}_{\eta}^2 = \xi g^2 \hat{\phi}^2 - (H' + H^2) \]

\[ \hat{m}_{A\theta}^2 = 2g(\partial_\tau - H)\hat{\phi} \]

\[ (\hat{m}^2)^{0i} = (\hat{m}^2)^{i0} = \frac{2}{\xi} H \partial^i \rightarrow \text{extra spatial derivative} \]
Third order diagram

\[ \frac{1}{2} \partial_\phi m_A^2(\tau_a) \int d^4x_b d^4x_c (\delta m^2)^{0i}(x_b)(\delta m^2)^{0j}(x_c) \sum_{\rho,a} \eta^{\mu\nu} D_{\mu\rho,ab} D_{\sigma\kappa,bc} D_{\tau\nu,ca} \]

= \ldots

= \ldots

= \partial_\phi m_A^2(\tau) H^2(\tau) \frac{-6(1 + \xi)}{64\pi^2\xi} \ln \Lambda^2
Take all diagrams together, go on shell
Final result: FLRW shifts scalar masses

\[ \Gamma = \int d^4 x \sqrt{-g} \left[ \frac{1}{2} \partial_{\mu} \phi_{cl} \partial^{\mu} \phi_{cl} - V(\phi_{cl}) \right. \\
- \frac{\Lambda^2}{16\pi^2} \left( \ddot{V}_{hh}(t) + \ddot{V}_{\theta\theta}(t) + 3g^2 \phi_{cl}(t)^2 \right) \\
+ \frac{\ln(\Lambda/\bar{m})}{32\pi^2} \left( \dddot{V}_{hh}(t) + \dddot{V}_{\theta\theta}(t) + 3g^4 \phi_{cl}(t)^4 - 6g^2 \phi_{cl}(t)^2 \ddot{V}_{\theta\theta}(t) \right) \left. \right] \\
\]

\[ \dddot{V}_{\alpha\alpha} \equiv V_{\alpha\alpha} - \dot{H} - 2H^2 \]
Initial conditions

\[ m^2(t) = \bar{m}^2 + \delta m^2(t) \]
\[ \delta m^2(0) = 0 \]

\[ \delta \phi(0) = (\delta \phi)'(0) \]
\[ = \mathcal{H}(0) = \mathcal{H}'(0) = 0 \]

- turns off off-diagonal mass terms at \( t=0 \)
- simplifies gauge field propagator
- connected to more realistic initial conditions by Bogolyubov transformation
Result consistent with

- Coleman & Weinberg ’74: time-independent case

- Shore ’80, Allen ’83, Ishikawa ’83: dS, Landau gauge

- Baacke, Heitmann ’96, ’97, ’99: time-dependent Minkowski

- Boyanovsky, Brahm, Holman, Lee, ’96: gauge invariance in Minkowski

- Garbrecht ’10: dS, static background field, $R_\xi$ gauge, adiabatic limit
Fermions

- Coleman-Weinberg

\[ \Gamma_{\text{1-loop (fermion)}} = -\frac{1}{16\pi^2} \sum_f \int d^3x dt \left[ m_\psi^2 \Lambda^2 - \frac{1}{4} m_\psi^4 \ln \Lambda^2 \right] \]

- time-dependent background (on shell)

\[ \Gamma_{\text{1-loop (fermion)}} = -\frac{1}{16\pi^2} \sum_f \int d^3x dt \left[ m_\psi^2 \Lambda^2 - \frac{1}{4} \left( m_\psi^4 - 2m_\psi^2 V_{\theta\theta} \right) \ln \Lambda^2 \right] \]

- FLRW (Yukawa mass, on shell)

\[ \Gamma_{\text{1-loop (fermion)}} = -\frac{1}{16\pi^2} \sum_f \int d^3x dt \sqrt{-g} \left[ m_\psi^2 \Lambda^2 - \frac{1}{4} \left( \left( m_\psi^2 + 2H^2 + \dot{H} \right)^2 - 2m_\psi^2 V_{\theta\theta} \right) \ln \Lambda^2 \right] \]
Conclusions

computed divergent part of the effective action for the Abelian U(1) Higgs model in FLRW

\[ \Gamma = \int d^4 x \sqrt{-g} \left\{ \frac{1}{2} \partial_\mu \phi_{cl} \partial^\mu \phi_{cl} - V(\phi_{cl}) \right. 
\]

\[ \left. - \frac{\Lambda^2}{16\pi^2} \left( \tilde{V}_{hh}(t) + \tilde{V}_{\theta\theta}(t) + 3g^2 \phi_{cl}(t)^2 \right) \right. \]

\[ \left. + \frac{\ln(\Lambda/\bar{m})}{32\pi^2} \left( \tilde{V}_{hh}^2(t) + \tilde{V}_{\theta\theta}^2(t) + 3g^4 \phi_{cl}(t)^4 - 6g^2 \phi_{cl}(t)^2 \tilde{V}_{\theta\theta}(t) \right) \right\} \]

showed manifest gauge invariance
Outlook: generalizations

- non-minimal coupling to gravity

- renormalize

- backreaction

- finite terms

- nonzero temperature