Elliptic integrable systems on the lattice and associated continuous systems

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Plan

1. Overview of integrable systems associated with an elliptic curve;
   - Continuous systems – integrable PDEs (LL, KN, Ell KdV, Ell KP);
   - Fully discrete systems ("lattice equations") – integrable PΔEs (discrete LL and KN).
2. Q4 and its soliton solutions;
3. General elliptic Lax system;
4. Elliptic lattice KdV and its continuous analogue;
5. Elliptic lattice KP and its analogue.
Landau-Lifschitz equation

This is a 2D PDE for a 3-component (unit) spin vector $S = (S_1(x, t), S_2(x, t), S_3(x, t))$, given by:

$$S_t = S \times S_{xx} + (J^2 S) \times S, \quad S^2 = S \cdot S = 1, \quad J = \text{diag}(J_1, J_2, J_3).$$

which can be viewed as a classical analogue of the XYZ Heisenberg spin model.

Lax pair (Sklyanin, 1979):

$$\partial_x \varphi = \frac{i}{2} \sum_{j=1}^3 w_j(u) S_j \sigma_j \varphi, \quad \partial_t \varphi = \frac{i}{2} \sum_{j=1}^3 \left( \frac{w_1 w_2 w_3}{w_j} S_j \sigma_j + w_j(u) (S \times S_x)_j \sigma_j \right) \varphi.$$

where $w_1 = \rho/sn(u; k), w_2 = \rho \text{cn}(u; k)/sn(u; k), w_3 = \rho \text{dn}(u; k)/sn(u; k)$ and

$$\rho = \sqrt{J_1^2 - J_3^2}, \quad k = \sqrt{(J_2^2 - J_2^2)/(J_1^2 - J_3^2)}.$$


A related equation is the anisotropic chiral field equations (I Cherednik, 1981):

$$S_{t'} = S \times (JT), \quad T_x = T \times (JS), \quad S^2 = T^2 = \text{const}.$$

where $T = (T_1(x, t), T_2(x, t), T_3(x, t))$ is another spin variable. Associated Lax pair is by supplementing the LL Lax operator with $\partial_{t'} \varphi = (i/2) \sum_{j=1}^3 w_j(u_0 - u) T_j \sigma_j \varphi.$
Krichever-Novikov equation

Introduced in the construction of rank 2 genus 1 periodic solutions of the KP equation\(^1\), the Krichever-Novikov equation reads:

\[
    u_t = \frac{1}{4} u_{xxx} + \frac{3}{8} \frac{r(u) - u_{xx}^2}{u_x},
\]

where \( r(u) \) is a quartic polynomial. Typically \( r(u) = 4u^3 - g_2u - g_3 \), in the Weierstrass case. In the original derivation, KN emerges in the form:

\[
    c_t = \frac{1}{4} c_{xxx} + \frac{3}{8} \frac{1 - c_{xx}^2}{c_x} - \frac{3}{2} \wp(2c) c_x^3,
\]

which is connected to the rational form by the identification \( u = \wp(c) \).

In the case of rank 3 genus 1 reductions of KP, Mokhov obtained a system which could be viewed as an elliptic version of the Boussinesq (BSQ) equation\(^2\). The construction, employs commuting differential operators together with the Lax pair of KP, yields a Lax pair for the reduced equations.

Whilst the Mokhov system has been little studied, for the KN equation some results have been established: infinite number of conservation laws (V Sokolov, 1984), bi-Hamiltonian structure (I Dorfman, 1987), bilinear form and algebro-geometric solution scheme (D Novikov, 1999), but until recently no explicit solutions. Elliptic soliton solutions were established together with the discrete analogue (Atkinson, Hietarinta & Nijhoff, 2007).

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The following system arises from the consideration of singular integral equations with an elliptic Cauchy kernel\(^3\)

\[
\begin{align*}
 s_t &= \frac{1}{4} s_{xxx} + \frac{3}{2} s_x \left[ R(s^2) - A^2 + A \frac{s_x}{s} - \frac{1}{2} \frac{s_{xx}}{s} \right] \\
 A_t &= \frac{1}{4} A_{xxx} - \frac{3}{2} A^2 A_x + \frac{3}{2} A_x R(s^2) + \frac{3}{4} \frac{s_x}{s} (R(s^2))_x
\end{align*}
\]

in which (for later reference) \( A = -u + w/s \), and where

\[
R(s^2) = \frac{1}{s^2} + 3e + gs^2,
\]

defines an elliptic curve: \( y^2 = R(x) \), which can be parametrized in Weierstrass elliptic functions.

Elliptic KP

A generalization of the KP equation associated with an elliptic curve was given in the context of discrete soliton equations\(^4\), but no discrete analogue was given. The elliptic KP takes the form:

\[
\begin{align*}
\left( u_t - \frac{1}{4} u_{xxx} - \frac{3}{2} u_x^2 + 3c^2 e^{v - \frac{v}{2}} \right)_x &= \frac{3}{4} u_{yy}, \\
v_t &= \frac{3}{2} v_x v_y - \frac{1}{2} v_{xxx} - \frac{1}{2} v_x^3 + \frac{3}{2} (a^2 + b^2) v_x + \frac{3}{2} [u_y + \overline{u}_y + v_x (\overline{u}_x + u_x)] \\
v_x &= \overline{u} - u . \quad c = (a^2 - b^2)/4
\end{align*}
\]

The associated curve is a Jacobi type curve: \(\omega^2 = (k^2 - a^2)(k^2 - b^2)\).

This is, in fact, a 3+1-dimensional system of PDEs for coupled dependent variables \(u(x, y, t; n)\), \(v(x, y, t; n)\) with one discrete variable \(n\), s.t. \(\overline{u} = u(x, y, t; n + 1)\), \(\overline{v} = v(x, y, t; n + 1)\) and \(u = u(x, y, t; n - 1)\), \(v = v(x, y, t; n - 1)\).

A bilinear scheme was given (a construction in terms of fermionic representation) and associated soliton solutions. The construction is closely connected to that of the LL equation.

Lax pair:

\[
\begin{align*}
\left[ \partial_x^2 + v_x \partial_x + \frac{1}{2} (v_{xx} - v_y + 3u_x - (a^2 + b^2)) \right] \varphi &= c \left( e^{v - \frac{v}{2}} \varphi + \varphi \right) \\
\left[ \partial_y + \partial_x^2 + 2u_x - \frac{1}{2} (a^2 + b^2) \right] \varphi &= 2ce^{v - \frac{v}{2}} \varphi \\
\left[ \partial_t - \partial_x^3 - 3u_x \partial_x + \frac{3}{2} (u_y - u_{xx}) \right] \varphi &= 3ce^{v - \frac{v}{2}} (u - \overline{u}) \varphi .
\end{align*}
\]

Lattice versions of LL equations

Spin discretization A lattice discretization of the LL as a spin system was given\(^5\), based on a discrete Lax pair:

\[
\tilde{\varphi} = S_0 \varphi + i \sum_{j=1}^{3} w_j(u) S_j \sigma_j, \quad \tilde{\varphi} = T_0 \varphi + i \sum_{j=1}^{3} w_j(u_0 - u) T_j \sigma_j,
\]

for two spin vectors \(S\) and \(T\), in combination with scalar functions \(S_0, T_0\).

Notation for functions \(f = f(n, m)\) of discrete variables \(n, m \in \mathbb{Z}\):

\[
\tilde{f} = f(n + 1, m), \quad \hat{f} = f(n, m + 1).
\]

Using the addition formula:

\[
w_j(u) w_k(u_0 - u) = w_i(u) w_k(u_0) + w_i(u_0 - u) w_j(u_0), \quad i, j, k = 1, 2, 3 \text{ cyclic},
\]

and setting \(J_i := w_i(u_0)\), we obtain the system

\[
T_0 \tilde{S} - \tilde{S} \times (JT) = \tilde{T}_0 S - (J\tilde{T}) \times S,
\]

\[
S_0 \tilde{T} - \tilde{T} \times (JS) = \tilde{S}_0 S - (J\tilde{S}) \times T,
\]

\[
\tilde{S}_0 T_0 - \tilde{T}_0 S_0 = J_1 J_2 J_3 \left( \tilde{S} \cdot (J^{-1} T - \tilde{T} \cdot (J^{-1} S)) \right),
\]

together with \(\tilde{S} \cdot (JT) = \tilde{T} \cdot (JS)\). Using the Casimirs \(S^2, T^2, S_0^2 + (JS)^2\) and \(T_0^2 + (JT)^2\) the spin \(T\) can be eliminated and a lattice equation in terms of \(S\) alone can be derived, which constitutes a lattice version of the LL equation.

Another lattice version of LL was presented\textsuperscript{6} using a stereographic projection, i.e. in terms of a coupled set of fields $u_k(n), v_k(n)$ over a chain labelled by $k \in \mathbb{Z}$ and “update”-variable $n$ with $\tilde{u}_k(n) = u_k(n+1)$, $\tilde{v}_k(n) = v_k(n+1)$:

\[
\tilde{u}_k = \frac{K u_k - L}{M u_k + N}, \quad \tilde{v}_k = \frac{K v_k + L}{-M v_k + N},
\]

in which

\[
\begin{align*}
K - N &= 2c^{(2)} u_{k-1} v_{k+1} - (ac^{(1)} + c^{(3)})(u_{k-1} - v_{k+1}) - 2ac^{(2)} - 4bc^{(1)}, \\
K + N &= (u_{k-1} + v_{k+1})[\gamma_k \gamma_{k-1}(ac^{(0)} + 3c^{(2)}) + 4bc^{(1)} + 3ac^{(2)} + c^{(4)})]/(\gamma_{k-1} - \gamma_k) \\
L &= c^{(3)} u_{k-1} v_{k+1} + (ac^{(2)} + 2bc^{(1)})(u_{k-1} - v_{k+1}) + 4bc^{(2)} - a^2c^{(1)}, \\
M &= c^{(1)} u_{k-1} v_{k+1} + c^{(2)}(u_{k-1} - v_{k+1}) - c^{(3)}.
\end{align*}
\]

Here $c^{(j)} = \beta_k \gamma_{k-1}^{j-1} + \beta_{k-1} \gamma_k^{j-1}$ and the $(\beta_k, \gamma_k)$ are points on the elliptic curve $\mu^2 + R(\lambda) = 0$, where $R(u) = u^3 + au + b$, and are constrained by the conditions

\[
\tilde{\beta}_{k-1} = \beta_k, \quad \tilde{\beta}_{k} = \beta_{k-1}, \quad \tilde{\gamma}_{k-1} = \gamma_k, \quad \tilde{\gamma}_{k} = \gamma_{k-1}.
\]

The system carries a Lax representation which comes from a Darboux realisation of a dressing chain. It was obtained by considering coupled fractional linear transformations on the projective realisation of the LL equation.

Connection with (Sklyanin’s) semi-discretization

In his seminal paper\(^7\) Sklyanin introduced a semi-discrete version of the LL equation.

This chain can be defined by the Hamiltonian:

\[
H = \sum_{n \in \mathbb{Z}} \log \left\{ S_0(n)S_0(n + 1) + \sum_{i=1}^{3} \left( \frac{c_i}{c_0} - J_i \right) S_i(n)S_i(n + 1) \right\},
\]

together with the (quadratic) Poisson brackets:

\[
\{S_i(n), S_0(m)\} = (J_j - J_k)S_j(n)S_k(n)\delta_{n,m}, \quad \{S_i(n), S_j(m)\} = -S_0(n)S_k(n)\delta_{n,m},
\]

\((i, j, k = 1, 2, 3 \text{ cycl.})\). In stereographic projective coordinates one can write

\[
S(u, \nu) = \frac{1}{u - \nu} \left( 1 - uv, i(1 + uv), u + \nu \right),
\]

which allows the association with corresponding NLS type models, e.g. the Shabat-Yamilov chain. In this context Adler derived\(^8\) some “doubly discrete equations”. A particular equation given is the following 5-point scalar equation:

\[
\frac{a_2(v_n)v_{n+1}u_n + a_1(v_n)(v_{n+1} + u_n) + a_0(v_n)}{u_n - v_{n+1}} = \frac{a_2(v_n)v_{n-1}\tilde{v}_n + a_1(v_n)(v_{n-1} + \tilde{v}_n) + a_0(v_n)}{v_{n-1} - \tilde{v}_n},
\]

in which \(\tilde{u}_n = v_n\). This equation arises as a permutability condition for a Bäcklund type transformation of the Shabat-Yamilov chain.

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Discrete Krichever-Novikov equation (Q4)

Bäcklund transformations for the KN equation were found\(^9\) to be of the form

\[
u_x \tilde{u}_x = \frac{1}{A} H(u, \tilde{u}, a), \quad (a, A) \in \Gamma
\]

where

\[
H(x, y, z) \equiv \left( xy + xz + yz + \frac{g_2}{4} \right)^2 - (x + y + z)(4xyz - g_3)
\]

is the canonical biquadratic associated with a Weierstrass curve

\[
\Gamma : A^2 = 4a^3 - g_2a - g_3.
\]

The permutability condition for the BT to commute with another BT:

\[
u_x \hat{u}_x = \frac{1}{B} H(u, \hat{u}, b), \quad (b, B) \in \Gamma,
\]

(where again \((b, B) = (\wp(\beta), \wp'(\beta))\) is a point on the curve leads to the PΔE:

\[
k_0 u\hat{u}\tilde{u}\hat{u} - k_1 \left( u\hat{u}\hat{u} + u\hat{u}\hat{u} + u\hat{u}\hat{u} + u\hat{u}\hat{u} \right) + k_2 \left( \hat{u}u + u\tilde{u} \right)
\]

\[
- k_3 \left( u\tilde{u} + \hat{u}\tilde{u} \right) - k_4 \left( u\hat{u} + \hat{u}\tilde{u} \right) + k_5 \left( u + \hat{u} + \tilde{u} + \bar{u} \right) + k_6 = 0.
\]

with the explicit parametrisation:

\[
k_0 = A + B, \quad k_1 = aB + bA, \quad k_2 = a^2B + b^2A, \quad k_3 = \frac{AB(A + B)}{2(b - a)} - b^2A + B \left( 2a^2 - \frac{g_2}{4} \right)
\]

\[
k_4 = \frac{AB(A + B)}{2(a - b)} - a^2B + A \left( 2b^2 - \frac{g_2}{4} \right), \quad k_5 = \frac{g_3}{2} k_0 + \frac{g_2}{4} k_1, \quad k_6 = \frac{g_2}{16} k_0 + g_3 k_1
\]

identifying

\[
u = u_{n,m}, \quad \tilde{u} = u_{n+1,m}, \quad \hat{u} = u_{n,m+1}, \quad \bar{u} = \hat{u} = u_{n+1,m+1}.
\]

Other forms of Q4 & Solitons

There are various alternative forms for this equation, which was renamed Q4 in the classification of scalar affine-multilinear quad-graph equations\(^{10}\). The following ("Jacobi") form is due to Hietarinta\(^ {11}\)

\[
p(\hat{u}\tilde{u} + \tilde{u}\hat{u}) - q(u\hat{u} + \hat{u}u) - \frac{pQ - qP}{1 - p^2q^2} \left[(\hat{u}\tilde{u} + u\hat{u}) - pq(1 - u\tilde{u}\hat{u}\tilde{u})\right] = 0
\]

where \(p = (p, P)\), \(q = (q, Q)\) are points on the Jacobi curve \(\Gamma : X^2 = x^4 - (k + 1/k)x^2 + 1\). For this form explicit \(N\)-soliton solutions were constructed\(^ {12}\). They are of the form: \(u = f/g\), in which:

\[
f = 1 - \sum_{i=1}^{N} \frac{\Theta(\xi_1 + 2\lambda_i; k_*)}{\Theta(\xi_*; k_*)} \rho_i + \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{\Theta(\xi_1 + 2\lambda_i + 2\lambda_j; k_*)}{\Theta(\xi_*; k_*)} \rho_i \rho_j X_{ij}^2 - \sum_{i=1}^{N} \sum_{j=i+1}^{N} \sum_{k=j+1}^{N} \frac{\Theta(\xi_1 + 2\lambda_i + 2\lambda_j + 2\lambda_k; k_*)}{\Theta(\xi_*; k_*)} \rho_i \rho_j \rho_k X_{ij}^2 X_{ik}^2 X_{jk}^2 + \ldots,
\]

\[
g = 1 - \sum_{i=1}^{N} \frac{\Theta(\xi_1 + 2\lambda_i; k_*)}{\Theta(\xi_*; k_*)} \rho_i + \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{\Theta(\xi_1 + 2\lambda_i + 2\lambda_j; k_*)}{\Theta(\xi_*; k_*)} \rho_i \rho_j X_{ij}^2 - \sum_{i=1}^{N} \sum_{j=i+1}^{N} \sum_{k=j+1}^{N} \frac{\Theta(\xi_1 + 2\lambda_i + 2\lambda_j + 2\lambda_k; k_*)}{\Theta(\xi_*; k_*)} \rho_i \rho_j \rho_k X_{ij}^2 X_{ik}^2 X_{jk}^2 + \ldots.
\]


Where we have introduced the factors

\[ X_{ij} := \left( \frac{l_j^* l_i - l_i^* l_j}{l_j^* l_i + l_i^* l_j} \right) \Theta(\lambda_i^* - \lambda_j^*; k_*) \]

\[ = \left( \frac{l_j^* l_i - l_i^* l_j}{l_j^* l_i + l_i^* l_j} \right) \frac{H(\lambda_i^* - \lambda_j^*; k_*)}{H(\lambda_i^* + \lambda_j^*; k_*)} \]

and the plane-wave factors \( \rho_i \) depend on the discrete variables in the following way:

\[ \tilde{\rho}_i = \left( \frac{p_i^* l_i - p l_i^*}{p_i^* l_i + p l_i^*} \right) \Theta(\lambda_i^* - \alpha_*; k_*) \rho_i , \]

\[ \hat{\rho}_i = \left( \frac{q_i^* l_i - q l_i^*}{q_i^* l_i + q l_i^*} \right) \Theta(\lambda_i^* + \beta_*; k_*) \rho_i . \]

An essential feature of these solutions is that they involve a correspondence between
the original Jacobi elliptic curve \( \Gamma \) and another Jacobi curve \( \Gamma^* \) with modulus \( k_* \),
given by:

\[ k_* + \frac{1}{k_*} = 2 \frac{1 - T}{t^2} , \quad (t, T) \in \Gamma , \]

and where the new lattice parameters are given by the correspondence:

\[ \Delta := \left\{ ((p, P), (p_*, P_*)) \in \Gamma \times \Gamma_* \mid p_*^2 = p \frac{pT - tP}{1 - p^2 t^2} , P_* = \frac{1}{t} \left( p - \frac{pT - tP}{1 - p^2 t^2} \right) \right\} . \]

where \( t = (t, T) \in \Gamma \). The correspondence \( \Delta \) is not a bijection, but the Abelian
group structures on \( \Gamma \) and \( \Gamma_* \) endow this set with a structure of an Abelian 2-group
(in the sense of Buchstaber and Veselov).
Introduce

\[ W(x) = \Phi_\omega(x)e^{-\eta x}, \quad \eta = \zeta(\omega), \quad \Phi_\kappa(x) = \frac{\sigma(x + \kappa)}{\sigma(x)\sigma(\kappa)}, \]

with \( \omega \) a half-period. We have the following identity (for general \( X, Y, Z \))

\[
(X - W(\xi + \alpha))(Y - W(\xi - \beta))(Z - W(\xi - \alpha + \beta))
- t(X - W(\xi - \alpha))(Y - W(\xi + \beta))(Z - W(\xi + \alpha - \beta)) =
\]

\[
= s \left[ W(\alpha)(Y W(\xi) + XZ) - W(\beta)(X W(\xi) + YZ)
+ \frac{W(\alpha)W(\beta)}{W(\alpha - \beta)} (Z W(\xi) + XY) - \frac{1}{W(\alpha - \beta)} (g + XYZ W(\xi)) \right],
\]

where

\[
t := \frac{\sigma(\xi - \alpha)\sigma(\xi + \beta)\sigma(\xi + \alpha - \beta)}{\sigma(\xi + \alpha)\sigma(\xi - \beta)\sigma(\xi - \alpha + \beta)}, \quad s := (t - 1)\frac{W(\alpha - \beta)}{W(\xi)}.
\]

The proof uses the addition formula:

\[
W(\xi)W(\xi') = W(\xi + \xi') \left[ \zeta(\xi) + \zeta(\xi') + \eta - \zeta(\xi + \xi' + \omega) \right], \quad W(-\xi) = -W(\xi).
\]

and the set of identities:

\[
t = \frac{W(\xi + \beta - \alpha)W(\xi) - W(\alpha)W(\beta)}{W(\xi + \alpha - \beta)W(\xi) - W(\alpha)W(\beta)} = \frac{W(\xi - \beta)W(\xi) - W(\alpha - \beta)W(\alpha)}{W(\xi + \beta)W(\xi) - W(\alpha - \beta)W(\alpha)}
\]

\[
= \frac{W(\xi + \beta - \alpha)W(\xi - \beta) - W(\alpha - \beta)W(\beta)}{W(\xi + \alpha - \beta)W(\xi + \beta) - W(\alpha - \beta)W(\beta)} = \frac{W(\xi + \alpha)W(\xi - \beta) + W(\alpha)W(\beta)}{W(\xi - \alpha)W(\xi + \beta) + W(\alpha)W(\beta)}
\]

\[
= \frac{g + W(\xi + \alpha)W(\xi - \beta)W(\xi + \beta - \alpha)W(\xi)}{g + W(\xi - \alpha)W(\xi + \beta)W(\xi - \beta + \alpha)W(\xi)}.
\]
The identity suggests the following homogeneous form for $Q_4$:

$$P(U\hat{U} + \tilde{U}\tilde{U}) - Q(U\tilde{U} + \hat{U}\tilde{U}) + \frac{P^2 - Q^2}{p + q} \left[ (U\hat{U} + \tilde{U}\tilde{U}) - \frac{1}{PQ}(g + U\tilde{U}\tilde{U}) \right] = 0,$$

where $P = W(\alpha)$, $Q = W(\beta)$, $U = W(\xi)$, on the curve:

$$p^2 = P^2 + 3e + gP^{-2}, \quad e = \varphi(\omega), \quad g = (e - e')(e - e'').$$

The corresponding 3-leg form follows from the identity:

$$t = \frac{(W(\tilde{\xi}) - W(\xi + \alpha))(W(\tilde{\xi}) - W(\xi - \beta))(W(\tilde{\xi}) - W(\xi - \alpha + \beta))}{(W(\tilde{\xi}) - W(\xi - \alpha))(W(\tilde{\xi}) - W(\xi + \beta))(W(\tilde{\xi}) - W(\xi + \alpha - \beta))} = \frac{\sigma(\xi - \alpha)\sigma(\xi + \beta)\sigma(\xi + \alpha - \beta)}{\sigma(\xi + \alpha)\sigma(\xi - \beta)\sigma(\xi - \alpha + \beta)}.$$
**General Elliptic Lax scheme**

Consider a lattice Lax pair of the form:

\[ \tilde{\chi}_\kappa = L_\kappa \chi_\kappa , \quad \hat{\chi}_\kappa = M_\kappa \chi_\kappa , \]

defining horizontal and vertical shifts of the vector function \( \chi_\kappa \), according to the diagram:

![Diagram showing shifts of \( \chi \) with matrices \( L \) and \( M \).]

with compatibility condition:

\[ \hat{L}_\kappa M_\kappa = \tilde{M}_\kappa L_\kappa \]

where the vectors \( \chi \) are located at the vertices of the quadrilateral and in which the matrices \( L \) and \( M \) are attached to the edges linking the vertices.

Assume form of \( N \times N \) matrices:

\[
\begin{align*}
(L_\kappa)_{i,j} & = \Phi_{N\kappa}(\tilde{\xi}_i - \xi_j - \alpha) h_j , \\
(M_\kappa)_{i,j} & = \Phi_{N\kappa}(\hat{\xi}_i - \xi_j - \beta) k_j ,
\end{align*}
\]

\( (i,j = 1, \ldots, N) \) where \( \Phi_\kappa(x) := \frac{\sigma(x + \kappa)}{\sigma(x) \sigma(\kappa)} \)

in which \( \xi = \xi_{n,m} \) are the main dependent variables, while the coefficients \( h_j, k_j \) are functions of the variables \( \xi_l \), that remain to be determined.
Working out the compatibility condition:

\[ \hat{L}_\kappa M_\kappa = \tilde{M}_\kappa L_\kappa , \]

using the addition formula

\[ \Phi_\kappa(x)\Phi_\kappa(y) = \Phi_\kappa(x+y) \left[ \zeta(\kappa) + \zeta(x) + \zeta(y) - \zeta(\kappa + x + y) \right] , \]

the consistency gives rise to

\[
\sum_{l=1}^{N} \hat{h}_l k_j \left[ \zeta(\hat{\xi}_i - \hat{\xi}_l - \alpha) + \zeta(\hat{\xi}_l - \xi_j - \beta) + \zeta(N\kappa) - \zeta(N\kappa + \hat{\xi}_i - \xi_j - \alpha - \beta) \right] = \\
= \sum_{l=1}^{N} \tilde{k}_l h_j \left[ \zeta(\hat{\xi}_i - \tilde{\xi}_l - \beta) + \zeta(\tilde{\xi}_l - \xi_j - \alpha) + \zeta(N\kappa) - \zeta(N\kappa + \hat{\xi}_i - \xi_j - \alpha - \beta) \right] \\
\quad (i,j = 1, \ldots, N) .
\]

Due to the dependence on the spectral parameter \( \kappa \) these equations separate into two parts:

\[
\left( \sum_{l=1}^{N} \hat{h}_l \right) k_j = \left( \sum_{l=1}^{N} \tilde{k}_l \right) h_j \quad (j = 1, \ldots, N) , \\
\sum_{l=1}^{N} \hat{h}_l \left[ \zeta(\hat{\xi}_i - \hat{\xi}_l - \alpha) + \zeta(\hat{\xi}_l - \xi_j - \beta) \right] k_j = \sum_{l=1}^{N} \tilde{k}_l \left[ \zeta(\hat{\xi}_i - \tilde{\xi}_l - \beta) + \zeta(\tilde{\xi}_l - \xi_j - \alpha) \right] h_j \\
\quad (i,j = 1, \ldots, N) .
\]
There are two nontrivial scenarios which we refer to as “Landau-Lifschitz type” and “Krichever-Novikov type” respectively:

1. Discrete Landau-Lifschitz (LL) type case: $\sum_l h_l \neq 0$, in which case we have that the variables $h_j, k_j$ are proportional to each other, $k_j = \rho h_j$, and after summation we obtain the conservation law:

$$\frac{\sum_{l=1}^N \hat{h}_l}{\sum_{l=1}^N h_l} = \frac{\sum_{l=1}^N \hat{k}_l}{\sum_{l=1}^N k_l}.$$  

and in which case the Lax eqs. reduce to:

$$\sum_{i=1}^N \left[ \zeta (\hat{\xi}_i - \hat{\xi}_i - \alpha) \rho \hat{h}_i - \zeta (\hat{\xi}_i - \hat{\xi}_j - \beta) \hat{k}_i \right] = \sum_{i=1}^N \left[ \zeta (\xi_j - \hat{\xi}_i + \beta) \rho \hat{h}_i - \zeta (\xi_j - \hat{\xi}_i + \alpha) \hat{k}_i \right]$$

$(i, j = 1, \ldots, N)$. It is worth mentioning that period-1 reduction (i.e., $\hat{\chi} = \lambda \chi$) reduces the corresponding PDEs to a finite-dimensional system of ODEs which are constitute the discrete-time Ruijsenaars system.\(^{13}\) We will assume that the centre of mass $\Xi = \sum_{j=1}^N \xi_j$ obeys the relation $\hat{\Xi} + \hat{\Xi} = \hat{\Xi} + \Xi$.

2. Krichever-Novikov (KN) type case: $\sum_l h_l = \sum_l k_l = 0$, in which case the first of the separated Lax relations becomes vacuous. In this case we will reduce the system further by setting $\sum_l \xi_l = 0$ (modulo the period lattice of the elliptic fuctions), and by assuming the proportionality $k_j = \rho h_j$.

Resolution of the compatibility conditions

Under the assumption

\[ \tilde{\Xi} + \hat{\Xi} = \hat{\Xi} + \Xi , \quad \text{where} \quad \Xi := \sum_{j=1}^{N} \xi_j \]

we can analyse the Lax eqs. by considering the following elliptic rational function:

\[
F(\xi) := \prod_{l=1}^{N} \frac{\sigma(\xi - \hat{\xi}_l)\sigma(\xi - \xi_l - \alpha - \beta)}{\sigma(\xi - \hat{\xi}_l - \alpha)\sigma(\xi - \tilde{\xi}_l - \beta)}
\]

\[ = \sum_{l=1}^{N} \left[ \zeta(\xi - \hat{\xi}_l - \alpha) - \zeta(\eta - \hat{\xi}_l - \alpha) \right] H_l + \sum_{l=1}^{N} \left[ \zeta(\xi - \tilde{\xi}_l - \beta) - \zeta(\eta - \tilde{\xi}_l - \beta) \right] K_l \]

which holds as an identity for any four sets of variables \( \xi_l, \hat{\xi}_l, \tilde{\xi}_l, \hat{\xi}_l \) s.t. the above equality for the sum holds. Here \( \eta \) is any one of the zeroes (i.e., \( \hat{\xi}_i \) or \( \xi_i + \alpha + \beta \)).

The coefficients \( H_j, K_j \) are explicitly given by:

\[
H_l = \frac{\prod_{k=1}^{N} \sigma(\hat{\xi}_l - \hat{\xi}_k + \alpha)\sigma(\hat{\xi}_l - \xi_k - \beta)}{\left[ \prod_{k=1}^{N} \sigma(\hat{\xi}_l - \tilde{\xi}_k - \gamma) \right] \prod_{k \neq l} \sigma(\hat{\xi}_l - \tilde{\xi}_k)} ,
\]

\[
K_l = \frac{\prod_{k=1}^{N} \sigma(\tilde{\xi}_l - \hat{\xi}_k + \beta)\sigma(\tilde{\xi}_l - \xi_k - \alpha)}{\left[ \prod_{k=1}^{N} \sigma(\tilde{\xi}_l - \hat{\xi}_k + \gamma) \right] \prod_{k \neq l} \sigma(\tilde{\xi}_l - \hat{\xi}_k)} .
\]

The coefficients obey the identity: \( \sum_{l=1}^{N} (H_l + K_l) = 0 \).
The generalized lattice systems

Using the identities above, taking $\xi = \hat{\xi}_i$, $\eta = \xi_j + \alpha + \beta$ in $F(\xi)$, and comparing with the Lax equations, we can identify:

$$tH_l = \rho \hat{h}_l, \quad -tK_l = \tilde{k}_l = \tilde{\rho} \hat{h}_l, \quad l = 1, \ldots, N,$$

with $t$ an arbitrary proportionality factor. Thus, inserting the explicit expressions for $H_l$ and $K_l$ we obtain a system of $N + 2$ equations for the $N + 2$ unknowns: $\xi_1, \ldots, \xi_N, \rho, t$. This comprises the set of equations

$$\tilde{t} \tilde{H}_l + \frac{t}{\tilde{\rho}} \tilde{K}_l = 0, \quad l = 1, \ldots, N$$

$\tilde{\Xi} + \tilde{\Xi} = \hat{\Xi} + \Xi$.

**LL Case:** We can separate here centre of mass motion (i.e. dynamics for $\Xi$), and we remain with $N + 1$ equations (including the conservation law) for $N + 1$ unknowns ($\rho$, $t$ and N-1 variables $\xi_i$ modulo the centre of mass).

**KN Case:** In this case we set $\sum_{l=1}^N H_l = \sum_{l=1}^N K_l = 0$, and supplement it with the condition $\Xi = \sum_{l=1}^N \xi_l = 0$ (mod period lattice). In that case we have $N + 1$ variables (restricted by this condition, with CoM dynamics trivial) and $N+1$ equations, including the sum restriction for $H_l$ and $K_l$. 
Explicit example: Adler's lattice equation (discrete Krichever-Novikov eq.)

This is the case $N = 2$, in which case the Lax pair has the form:

$$
\tilde{\chi} = L_\kappa \chi = \lambda \begin{pmatrix}
\Phi_{2\kappa}(\xi - \xi - \alpha) & -\Phi_{2\kappa}(\xi + \xi - \alpha) \\
\Phi_{2\kappa}(-\xi - \xi - \alpha) & -\Phi_{2\kappa}(-\xi + \xi - \alpha)
\end{pmatrix} \chi
$$

$$
\tilde{\chi} = M_\kappa \chi = \mu \begin{pmatrix}
\Phi_{2\kappa}(\xi - \xi - \beta) & -\Phi_{2\kappa}(\xi + \xi - \beta) \\
\Phi_{2\kappa}(-\xi - \xi - \beta) & -\Phi_{2\kappa}(-\xi + \xi - \beta)
\end{pmatrix} \chi,
$$

in which the coefficients $\lambda$ and $\mu$ are functions $\lambda = \lambda(\xi, \xi; \alpha)$ and $\mu = \mu(\xi, \xi; \beta)$, respectively. The Lax equations are of the form:

$$
\hat{\lambda}_\mu \left[ \zeta(\xi - \xi - \alpha) + \zeta(\xi - \xi - \beta) - \zeta(\xi + \xi - \alpha) + \zeta(\xi + \xi + \beta) \right]
$$

$$
= \tilde{\mu} \lambda \left[ \zeta(\xi - \xi - \beta) + \zeta(\xi - \xi - \alpha) - \zeta(\xi + \xi - \beta) + \zeta(\xi + \xi + \alpha) \right]
$$

$$
\hat{\lambda}_\mu \left[ \zeta(\xi - \xi - \alpha) + \zeta(\xi + \xi - \beta) - \zeta(\xi + \xi - \alpha) + \zeta(\xi - \xi + \beta) \right]
$$

$$
= \tilde{\mu} \lambda \left[ \zeta(\xi - \xi - \beta) + \zeta(\xi + \xi - \alpha) - \zeta(\xi + \xi - \beta) + \zeta(\xi + \xi - \alpha) \right]
$$

$$
\hat{\lambda}_\mu \left[ \zeta(-\xi - \xi - \alpha) + \zeta(\xi - \xi - \beta) - \zeta(-\xi + \xi - \alpha) + \zeta(\xi + \xi + \beta) \right]
$$

$$
= \tilde{\mu} \lambda \left[ \zeta(\xi - \xi - \beta) + \zeta(\xi - \xi - \alpha) - \zeta(\xi + \xi - \beta) + \zeta(\xi + \xi + \alpha) \right]
$$

$$
\hat{\lambda}_\mu \left[ \zeta(-\xi - \xi - \alpha) + \zeta(\xi + \xi - \beta) - \zeta(-\xi + \xi - \alpha) + \zeta(\xi - \xi + \beta) \right]
$$

$$
= \tilde{\mu} \lambda \left[ \zeta(\xi - \xi - \beta) + \zeta(\xi - \xi - \alpha) - \zeta(\xi + \xi - \beta) + \zeta(\xi + \xi + \alpha) \right]
$$
We, thus, obtain:

\[
\hat{\lambda}_\mu = \frac{\sigma(2\tilde{\xi}) \sigma(\tilde{\xi} + \xi + \beta - \alpha)}{\sigma(\tilde{\xi} - \xi - \alpha) \sigma(\tilde{\xi} + \tilde{\xi} - \alpha) \sigma(\tilde{\xi} - \xi - \beta) \sigma(\tilde{\xi} + \xi + \beta)}
\]

\[
= \tilde{\mu}_\lambda \frac{\sigma(2\tilde{\xi}) \sigma(\tilde{\xi} + \xi + \alpha - \beta)}{\sigma(\tilde{\xi} - \xi - \beta) \sigma(\tilde{\xi} + \tilde{\xi} - \beta) \sigma(\tilde{\xi} - \xi - \alpha) \sigma(\tilde{\xi} + \xi + \alpha)}
\]

\[
\hat{\lambda}_\mu = \frac{\sigma(2\tilde{\xi}) \sigma(\tilde{\xi} - \xi + \beta - \alpha)}{\sigma(\tilde{\xi} - \xi + \alpha) \sigma(\tilde{\xi} + \tilde{\xi} + \alpha) \sigma(\tilde{\xi} - \xi - \beta) \sigma(\tilde{\xi} + \xi + \beta)}
\]

\[
= \tilde{\mu}_\lambda \frac{\sigma(2\tilde{\xi}) \sigma(\tilde{\xi} - \xi + \alpha - \beta)}{\sigma(\tilde{\xi} - \xi + \beta) \sigma(\tilde{\xi} + \tilde{\xi} + \beta) \sigma(\tilde{\xi} - \xi - \alpha) \sigma(\tilde{\xi} + \xi + \alpha)}
\]

\[
\hat{\lambda}_\mu = \frac{\sigma(2\tilde{\xi}) \sigma(\tilde{\xi} + \xi - \beta + \alpha)}{\sigma(\tilde{\xi} - \xi + \alpha) \sigma(\tilde{\xi} + \tilde{\xi} + \alpha) \sigma(\tilde{\xi} - \xi + \beta) \sigma(\tilde{\xi} + \xi + \beta)}
\]

\[
= \tilde{\mu}_\lambda \frac{\sigma(2\tilde{\xi}) \sigma(\tilde{\xi} - \xi - \alpha + \beta)}{\sigma(\tilde{\xi} - \xi + \beta) \sigma(\tilde{\xi} + \tilde{\xi} + \beta) \sigma(\tilde{\xi} - \xi + \alpha) \sigma(\tilde{\xi} + \xi - \alpha)}
\]
3-Leg equations

Eliminating the factors $\lambda \mu / (\tilde{\mu} \lambda)$ we get only two separate equations:

$$\frac{\sigma(\xi - \xi + \alpha) \sigma(\xi + \xi - \alpha)}{\sigma(\xi - \xi - \alpha) \sigma(\xi + \xi + \alpha)} \frac{\sigma(\xi - \xi - \beta) \sigma(\xi + \xi + \beta)}{\sigma(\xi - \xi + \beta) \sigma(\xi + \xi - \beta)} = \frac{\sigma(\xi - \xi - \gamma) \sigma(\xi + \xi + \gamma)}{\sigma(\xi - \xi + \gamma) \sigma(\xi + \xi - \gamma)}$$

and

$$\frac{\sigma(\tilde{\xi} - \tilde{\xi} + \alpha) \sigma(\tilde{\xi} + \tilde{\xi} + \alpha)}{\sigma(\tilde{\xi} - \tilde{\xi} - \alpha) \sigma(\tilde{\xi} + \tilde{\xi} - \alpha)} \frac{\sigma(\tilde{\xi} - \tilde{\xi} - \beta) \sigma(\tilde{\xi} + \tilde{\xi} + \beta)}{\sigma(\tilde{\xi} - \tilde{\xi} + \beta) \sigma(\tilde{\xi} + \tilde{\xi} + \beta)} = \frac{\sigma(\tilde{\xi} - \tilde{\xi} - \gamma) \sigma(\tilde{\xi} + \tilde{\xi} + \gamma)}{\sigma(\tilde{\xi} - \tilde{\xi} + \gamma) \sigma(\tilde{\xi} + \tilde{\xi} - \gamma)} .$$

Actually, they are one and the same equation, namely Adler’s lattice equation\(^{14}\). In rational form, setting $u = \wp(\xi)$, the equation can be rewritten as:

$$A \left[ (u - b)(\tilde{u} - b) - (a - b)(c - b) \right] \left[ (\tilde{u} - b)(\tilde{u} - b) - (a - b)(c - b) \right]$$

$$+ B \left[ (u - a)(\tilde{u} - a) - (b - a)(c - a) \right] \left[ (\tilde{u} - a)(\tilde{u} - a) - (b - a)(c - a) \right] =$$

$$= ABC(a - b)$$

The parameters $(a, A), (b, B)$ and $(c, C)$ in this equation are given points on the Weierstrass elliptic curve, i.e.

$$A^2 = r(a) \equiv 4a^3 - g_2a - g_3 \ , \quad B^2 = r(b) \ , \quad C^2 = r(c) ,$$

and can be parametrised in terms of the Weierstrass $\wp$-function:

$$(a, A) = (\wp(\alpha), \wp'(\alpha)), \quad (b, B) = (\wp(\beta), \wp'(\beta)), \quad (c, C) = (\wp(\beta - \alpha), \wp'(\beta - \alpha)).$$

These parameters are related through the addition formulae:

$$A(c - b) + B(c - a) = C(a - b) \ , \quad a + b + c = \frac{1}{4} \left( \frac{A + B}{a - b} \right)^2 .$$
Elliptic Lattice KdV

A 2-parameter generalisation of the lattice potential KdV equation associated with an elliptic curve was proposed\textsuperscript{15} which reads

\[
(p + q + u - \tilde{u}) (p - q + \hat{u} - \tilde{u}) = p^2 - q^2 + g (\tilde{s} - \hat{s}) (\tilde{s} - s)
\]

\[
\left[\left(p + u - \frac{\tilde{w}}{\tilde{s}}\right) \tilde{s} - \left(q + u - \frac{\hat{w}}{\hat{s}}\right) \hat{s}\right] \tilde{s} = \left[\left(p - \hat{u} + \frac{\hat{w}}{\hat{s}}\right) \hat{s} - \left(q - \hat{u} + \frac{\tilde{w}}{\tilde{s}}\right) \tilde{s}\right] s
\]

\[
\left(p - \tilde{u} + \frac{w}{s}\right) s + \left(q + \tilde{u} - \frac{\hat{w}}{\hat{s}}\right) \hat{s} \tilde{s} = \left[p + \hat{u} - \frac{\hat{w}}{\hat{s}}\right] \hat{s} + \left[q - \hat{u} + \frac{w}{s}\right] s \tilde{s}
\]

\[
\left(p + u - \frac{\tilde{w}}{\tilde{s}}\right) \left(p - \hat{u} + \frac{w}{s}\right) = p^2 - R(s\tilde{s})
\]

\[
\left(q + u - \frac{\hat{w}}{\hat{s}}\right) \left(q - \hat{u} + \frac{w}{s}\right) = q^2 - R(s\hat{s})
\]

associated with the elliptic curve:

\[
y^2 = R(x) = \frac{1}{x} + 3e + gx.
\]

Note:

If parm. \( g \rightarrow 0 \), the curve degenerates and the equation for \( u \) decouples \( \Rightarrow \) lattice (potential) KdV !

The elliptic lattice KdV was constructed on using “elliptic matrices” (i.e. matrices possessing a quasi-graded structure) and in terms of an elliptic Cauchy kernel. The following was established:

- internal consistency (well-posedness) of the system (from the IVP of view);
- the integrability (multidimensional consistency and Lax pair);
- existence of $N$-soliton solutions.

In fact, the system (1)-(5) is equivalent to a lattice ”correspondence” (i.e. multi-valued map in the sense of Veselov & Moser) in terms of variables $s$ and $A = -u + \frac{w}{s}$.

**Lax pair**

The Lax pair for the lattice system is given by:

\[ \tilde{\varphi} = L(K)\varphi, \quad \hat{\varphi} = M(K)\varphi \]

with matrices $L$ and $M$ are given by:

\[
L(K) = \begin{pmatrix}
p - \tilde{u} + \frac{g}{K}\tilde{w}s & 1 - \frac{g}{K}\tilde{s}s \\
K + 3e - p^2 + g\tilde{s}s & p + u - \frac{g}{K}\tilde{w}s \\
(p - \tilde{u})(p + u) + \frac{g}{K}\tilde{w}w & p + u - \frac{g}{K}\tilde{w}s
\end{pmatrix}
\]

and

\[
M(K) = \begin{pmatrix}
q - \hat{u} + \frac{g}{K}\hat{w}s & 1 - \frac{g}{K}\hat{s}s \\
K + 3e - q^2 + g\hat{s}s & q + u - \frac{g}{K}\hat{w}s \\
(q - \hat{u})(q + u) + \frac{g}{K}\hat{w}w & q + u - \frac{g}{K}\hat{w}s
\end{pmatrix}
\]

The discrete Lax equation

\[ \tilde{L}M = \hat{M}L \quad \Rightarrow \quad \text{Elliptic Lattice System} \]

**Note:** although the elliptic dependence in the spectral parameter is not manifest (compare with polynomial Lax pair for the LL equation found by Bordag and Yanovski) soliton type solutions depend on the elliptic curve.
Soliton type solutions

Introducing the $N \times N$ matrix $\mathcal{M}$ with entries

$$\mathcal{M}_{ij} = \frac{1 - g/(K_iK_j)}{k_i + k_j} \rho_i \quad (i, j = 1, \ldots, N)$$

Parameters of the solution $(k_i, K_i)$ are points on the elliptic curve:

$$k^2 = K + 3e + \frac{g}{K}$$

$$r = (r_i)_{i=1,\ldots,N}$$ vector with components

$$r_i = \left(\frac{p + k_i}{p - k_i}\right)^n \left(\frac{q + k_i}{q - k_i}\right)^m r_i^0$$

($r_i^0$ are independent of $n$, $m$).

and this leads to the following explicit formulae for the quantities of interest:

$$u = e \cdot (\mathcal{I} + \mathcal{M})^{-1} \cdot r$$

$$s = e \cdot K^{-1} \cdot (\mathcal{I} + \mathcal{M})^{-1} \cdot r$$

$$w = 1 + e \cdot K^{-1} \cdot (\mathcal{I} + \mathcal{M})^{-1} \cdot k \cdot r$$

in which have employed the vector $e = (1, 1, \ldots, 1)$ and the diagonal matrices

$$K = \text{diag}(K_1, K_2, \ldots, K_N) \quad , \quad k = \text{diag}(k_1, k_2, \ldots, k_N) .$$
An alternative elliptic KP system

The following is higher-dimensional generalization of the Elliptic KdV system \(^{16}\):

\[
\left( u_t - \frac{1}{4} u_{xxx} - \frac{3}{2} (u_x)^2 + \frac{3}{2} g s_x s'_x \right)_x = \frac{3}{4} u_{yy} + \frac{3}{2} g \left( s'_x s_y - s'_y s_x \right),
\]

\[
(s s')_t = \frac{1}{4} (s_{xxx} s' + s'_{xxx} s) + \frac{3}{2} u_x (s s')_x - 3 u s_x s'_x + \frac{3}{4} (s'_y s - s'_s - s_x s'_x)_x
\]

\[
+ \frac{3}{2} (s'_s s - s'_s s'_x + w'_y s - w_y s') + \frac{3}{2} (w_x s'_x + w'_x s_x),
\]

\[
(s s')_y = 2 s w'_x - 2 s' w_x + 2 u (s_x s'_x - s'_x s) + s_{xx} s' - s'_{xx} s,
\]

\[
\left( u + \frac{w}{s} \right)_x + \left( u - \frac{w}{s} \right)_x + \frac{w}{s} (u - u) = \left( \frac{1}{s' s} + 3 e + gss' \right) + U_{1,0} - U_{1,0},
\]

\[
\left( u + \frac{w'}{s'} \right)_x + \left( u - \frac{w'}{s'} \right)_x + \frac{w'}{s'} (u - u) = \left( \frac{1}{s' s} + 3 e + gss' \right) + U_{0,1} - U_{0,1},
\]

\[s' w = w' s.\]

This is to be seen as a coupled system in terms of \( u, s, s', w, w' \), depending on the continuous variables \( x, y, t \) and a discrete variable \( n \), and denoting shifts by

\[\bar{f} = f(x, y, t; n + 1)\]

In the above system \( U_{1,0} \) and \( U_{0,1} \) are to be eliminated by using in addition the relation

\[U_{1,0} + U_{0,1} = u_x + u^2 - gss',\]

but the present form makes clear that by imposing the reduction \( \bar{f} = f \) (for all dependent variables) together with \( s = s' \), \( w = w' \) as well as \( U_{1,0} = U_{0,1} \) we get a reduction to the continuous elliptic KdV system.

\(^{16}\) P. Jennings & FWN, *On an elliptic extension of the Kadomtsev-Petviashvili equation*, in prep.
Lax pair

This system is integrable in that it admits a Lax representation, given by

\[
\begin{align*}
\varphi_y &= \varphi_{xx} + A\varphi + B\varphi, \\
\varphi_x &= J\varphi + C\varphi + D\varphi, \\
\varphi_t &= \varphi_{xy} + E\varphi + F\varphi_x + G\varphi,
\end{align*}
\]

in which

\[
J = \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}, \quad
A = \begin{pmatrix}
2u_x & 0 \\
2(U_{1,0})_x & 0
\end{pmatrix}, \quad
B = \begin{pmatrix}
-2gs'_x w & 2gs'_x s \\
-2gw'_x w & 2gw'_x s
\end{pmatrix},
\]

and

\[
C = \begin{pmatrix}
-u & 1 \\
3e - U_{1,0} - \overline{U}_{0,1} & \frac{1}{u}
\end{pmatrix}, \quad
D = \begin{pmatrix}
gws' & -gss' \\
gww' & -gsw'
\end{pmatrix},
\]

and

\[
E = \begin{pmatrix}
u_y - (U_{0,1})_x - \frac{u(s'_y + v'_x)}{s'} & u_x + \frac{s'_y + v'_x}{s'} \\
(U_{1,0})_y - (U_{1,1})_x - \frac{u((U_{1,-1})_x - w'_y)}{s'} & (U_{1,0})_x + \frac{v'_y - (U_{1,-1})_x}{s'}
\end{pmatrix},
\]

and finally,

\[
F = \begin{pmatrix}
\frac{-s'_y - v'_x}{s'} & 0 \\
\frac{(U_{1,-1})_x - w'_y}{s'} & 0
\end{pmatrix}, \quad
G = \begin{pmatrix}
-gs'_x U_{1,-1} & -gs'_x v \\
-gw'_x U_{-1,1} & -gw'_x v
\end{pmatrix},
\]

where \( U_{1,-1} = (1 - \overline{v}w')/\overline{s} \) and \( U_{-1,1} = (1 - \overline{v}w')/s' \).
We consider now variables of 3+1 discrete variables: three variables $n$, $m$ and $l$ which are on the same footing (with associated lattice parameters $p$, $q$ and $r$ respectively), and a fourth discrete variable $N$ with which no parameter is associated.

The abbreviated notations for lattice shifts are: for $f = f(n, m, h; N)$ we have

\[
\begin{align*}
\tilde{f} &= f(n+1, m, l; N), \\
\hat{f} &= f(n, m+1, l; N), \\
\check{f} &= f(n, m, l+1; N), \\
\bar{f} &= f(n, m, l; N+1),
\end{align*}
\]

The lattice KP equation in 3 dimensions reads\textsuperscript{17}

\[
(p - \hat{u}) \left( q - r + \ddot{u} - \dddot{u} \right) + (q - \tilde{u}) \left( r - p + \ddot{u} - \dddot{u} \right) + (r - \check{u}) \left( p - q + \hat{u} - \dddot{u} \right) = 0,
\]

and this is related to the famous Hirota bilinear KP equation. The elliptic lattice system\textsuperscript{18} is a bit more complicated and lives in 3+1 dimensions.

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\textsuperscript{18}P Jennings & FWN, On an elliptic extension of the Kadomtsev-Petviashvili equation, in preparation.
Elliptic lattice KP system

\[(p - \tilde{u}) (q - r + \tilde{u} - \tilde{u}) + (q - \tilde{u}) (r - p + \tilde{u} - \tilde{u}) + (r - \tilde{u}) (p - q + \tilde{u} - \tilde{u}) = g \left( \tilde{s}' (\tilde{s} - \tilde{s}) + \tilde{s}' (\tilde{s} - \tilde{s}) + \tilde{s}' (\tilde{s} - \tilde{s}) \right), \]

\[
\frac{(p + \tilde{u}) \dot{s} - (q + \tilde{u}) \dot{s}}{\dot{s}} + \frac{(q + \tilde{u}) \dot{s} - (r + \tilde{u}) \dot{s} + \tilde{w} - \tilde{w}}{\dot{s}} + \frac{(r + \tilde{u}) \dot{s} - (p + \tilde{u}) \dot{s} + \tilde{w} - \tilde{w}}{\dot{s}} = 0, \quad \text{and cycl.}
\]

\[
\left( p + u - \frac{\tilde{w}}{\tilde{s}} \right) \left( p - \tilde{u} + \frac{w}{s} \right) - p^2 - \frac{\tilde{w}}{\tilde{s}} (\tilde{u} - \tilde{u}) + \left( \frac{1}{\tilde{s} \tilde{s}'} + 3e + g \tilde{s}' s \right)
\]

\[
= \left( \tilde{U}_{1,0} - \tilde{U}_{0,1} \right) - (U_{1,0} - U_{0,1}), \quad \text{and cycl.}
\]

\[s' \tilde{w} = w' \tilde{s},\]

Here, the quantities \( U_{1,0} \) and \( U_{0,1} \) are to be eliminated. We have in addition the following relation for these quantities:

\[\tilde{U}_{0,1} + U_{1,0} = p (\tilde{u} - u) + \tilde{u} u - g \tilde{s}' s, \quad \text{and cycl.}\]
The corresponding Lax representation takes the form of the triplet of equations:

\[
\begin{align*}
\tilde{\varphi} &= A_0 \varphi + A_1 \varphi + J\varphi, \\
\hat{\varphi} &= B_0 \varphi + B_1 \varphi + J\varphi, \\
\dot{\varphi} &= C_0 \varphi + C_1 \varphi + J\varphi,
\end{align*}
\]

where \( J = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \), and

\[
A_0 = \begin{pmatrix} p - \tilde{u} & 1 \\ 3e - \tilde{U}_{0,1} - \tilde{U}_{1,0} & p + \tilde{u} \end{pmatrix}, \quad A_1 = g \begin{pmatrix} -\tilde{s}' w & \tilde{s}' s \\ -\tilde{w}' w & \tilde{w}' s \end{pmatrix},
\]

and the matrices \( B \) and \( C \) similarly, replacing parameters and shifts.

Both the continuous elliptic KP system as well as the elliptic lattice system admit \((N, N')\)-soliton solutions, which can be written in the form of Cauchy matrix expansions, with Cauchy matrices defined in terms of an elliptic kernel.
Soliton Type Solutions

The soliton solutions are given in terms of an elliptic Cauchy kernel $M$ given by

$$M_{ij} = \frac{1 - g/K_iK_j'}{k_i + k_j'} \rho_i \sigma_j, \quad i = 1, \ldots, N, \quad j = 1, \ldots, N'$$

where the parameters of the solutions $(k_i, K_i)$ and $(k_j', K_j')$ are points on the elliptic curve

$$k^2 = K + 3e + \frac{g}{K}, \quad k'^2 = K' + 3e + \frac{g}{K'},$$

and where the entries $\rho_i$ and $\sigma_j$ depend on the dynamical variables as follows:

$$\rho_i = (1+k_i/p)^n(1+k/q)^m(1+k/r)^lK^{N}, \quad \sigma_j = (1-k'/p)^{-n}(1-k'/q)^m(1-k'/r)^{-l}(-K')^N.$$

Introducing column- and row-vectors

$$r = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_N \end{pmatrix}, \quad t{s} = (\sigma_1, \cdots, \sigma_{N'}),$$

and an arbitrary $N' \times N$ coefficient matrix $C$, defining the diagonal matrices

$$k = \text{diag}(k_1, k_2, \ldots, k_N), \quad K = \text{diag}(K_1, K_2, \ldots, K_N),$$

$$k' = \text{diag}(k'_1, k'_2, \ldots, k'_{N'}), \quad K' = \text{diag}(K'_1, K'_2, \ldots, K'_{N'}).$$

This leads to the following explicit solutions:

$$u = {t{s}}(I + CM)^{-1}Cr,$$

$$s = {t{s}}K^{-1}(I + CM)^{-1}Cr, \quad s' = {t{s}}(I + CM)^{-1}CK^{-1}r,$$

$$w = 1 + {t{s}}K'^{-1}(I + CM)^{-1}Kr, \quad w' = 1 + {t{s}}k'(I + CM)^{-1}K'^{-1}r.$$
Conclusions

- There exist two broad classes of elliptic (classical) lattice equations: LL (or spin non-zero), and KN (spin zero) type;
- Many results (in terms of structure and solutions) exist for Q4 and elliptic KdV, but very few for the various versions of LL type (none in terms of solutions);
- There exist also higher-dimensional lattice and continuous equations of elliptic type, but they necessarily live in four dimensions;
- A unifying framework for all these systems, and establishing connections with representation theory is highly desirable.