4d CFTs, Riemann surfaces, and elliptic integrable models: a 6d story

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The goal and the context

- Recently there was a lot of progress in computing partition functions of supersymmetric QFTs in various space-time dimensions.

- Some of these partition functions can be computed exactly which is a quite rare luxury.

- When this happens the partition functions usually reduce to finite dimensional matrix model integrals.

- The integrands are given in terms of special functions: elliptic Gamma functions in some of the 4d cases, hyperbolic Gamma functions in analogous 3d situations, etc.

- An example of such a partition function in 4d is the supersymmetric index, $S^3 \times S^1$, relation of which to the hypergeometric elliptic integrals was pointed out in a beautiful paper by Dolan and Osborn (2008) and was then further studied in a series of papers by Spiridonov and Vartanov (and others).

- The mathematical results regarding properties of these integrals (Rains, Spiridonov, · · ·) allow us to check, and in some cases give evidence for new, non trivial properties of supersymmetric QFTs.
The goal and the context: \(6 = 4 + 2\)

- Recently there was an unrelated dramatic development in understanding CFTs in 4d with extended supersymmetry, \(\mathcal{N} = 2\). (Gaiotto, Gaiotto-Moore-Neitzke, Argyres-Seiberg, ...)

- There exists a special 6d superconformal theory defined by its supersymmetry and some discrete data (ADE). In string/M-theory the \(A_{N-1}\)-type model of this kind is the theory living on \(N\) M5-branes.

- Compactifying this 6d theory down to four dimensions on a punctured Riemann surface one obtains a wide variety of 4d superconformal theories labeled by the choice of the Riemann surface. They are usually called theories of class \(S(i\chi)\).

- Some of these 4d theories are usual gauge theories but others are less conventional strongly-coupled SCFTs. Theories of class \(S\) are interrelated by a network of strong/weak coupling dualities and RG flows.

- One can compute some of the supersymmetric partition functions, eg the index, for this new class of theories (even for the strongly-coupled ones).

- Our goal in this talk will be to review some of the mathematics one encounters while computing some of the partition functions for theories of class \(S\).
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- Recently there was an unrelated dramatic development in understanding CFTs in 4d with extended supersymmetry, \( \mathcal{N} = 2 \). (Gaiotto, Gaiotto-Moore-Neitzke, Argyres-Seiberg, …)

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- Our goal in this talk will be to review some of the mathematics one encounters while computing some of the partition functions for theories of class \( S \).
Outline

- “$A_1$ symmetric province”
- “$A_{N-1}$ symmetric kingdom”
- “$A_1$ non-symmetric empire”
- Comments
Notations

- Elliptic Gamma function:
  \[
  \Gamma(z; p, q) = \prod_{i,j=0}^{\infty} \frac{1 - p^{i+1} q^{j+1} z^{-1}}{1 - p^i q^j z}.
  \]

- We use the short-hand notation
  \[
  f(a_0 a_1^{\pm 1} a_2^{\pm 1} \cdots) = \prod_{\alpha_i = \pm 1} f(a_0 a_1^{\alpha_1} a_2^{\alpha_2} \cdots).
  \]

- Theta function is given by
  \[
  \theta(z; q) = \prod_{\ell=0}^{\infty} (1 - q^\ell z)(1 - q^{1+\ell} z^{-1}).
  \]

- Let us also define
  \[
  \kappa \equiv \Gamma \left( \frac{p q}{t}; p, q \right) \prod_{\ell=1}^{\infty} (1 - q^\ell)(1 - p^\ell).
  \]

- We will always assume that
  \[
  |p|, |q|, \left| \frac{p q}{t} \right| < 1.
  \]

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$A_1$ symmetric Province
We are interested in a very specific class of functions which are labeled by the topological information of a punctured Riemann surface.

By topological information of a Riemann surface, $C_{g,s}$, we mean the genus and the number of punctures. We recursively define the following symmetric functions:

$$Z_{g,s}(a_1, a_2, \ldots, a_s; t, p, q).$$

Here symmetric means that the functions are invariant under inversions of any number of the $a_i$ arguments.

The three punctured sphere, $C_{0,3}$, corresponds to

$$Z_{0,3}(a, b, c; t, p, q) = \Gamma \left( t^{\frac{1}{2}} a^{\pm 1} b^{\pm 1} c^{\pm 1}; p, q \right).$$
Riemann surfaces → Integrals: the recursion

- Given the functions corresponding to two Riemann surfaces, $C_{g_1, s_1}$ and $C_{g_2, s_2}$, one can obtain the function corresponding to $C_{g_1+g_2, s_1+s_2-2}$ by “gluing” the two surfaces along a puncture.

\[
Z_{g_1+g_2, s_1+s_2-2}(a_1, \cdots, a_{s_1-1}, b_1, \cdots, b_{s_2-1}; t, p, q) = \kappa \times \int \frac{dz}{4\pi iz} \frac{\Gamma \left( \frac{p}{t} z^\pm 2; p, q \right)}{\Gamma \left( z^\pm 2; p, q \right)} Z_{g_1, s_1}(a_1, \cdots, a_{s_1-1}, z; t, p, q) Z_{g_2, s_2}(b_1, \cdots, b_{s_2-1}, z^{-1}; t, p, q).
\]

- Given the function corresponding to $C_{g, s}$ one can obtain the function corresponding to $C_{g+1, s-2}$ by gluing two punctures together:

\[
Z_{g+1, s-2}(a_1, \cdots, a_{s_1-2}; t, p, q) = \int \frac{dz}{4\pi iz} \frac{\Gamma \left( \frac{p}{t} z^\pm 2; p, q \right)}{\Gamma \left( z^\pm 2; p, q \right)} Z_{g, s}(a_1, \cdots, a_{s_1-2}, z, z^{-1}; t, p, q).
\]
Riemann surfaces $\rightarrow$ Integrals: consistency

- Thus given a Riemann surface $C_{g,s}$ one constructs a function corresponding to it recursively by decomposing the surface into pairs-of-pants and then gluing them together.

- In general, a given Riemann surface has different pairs-of-pants decompositions. So, is the recursive procedure well defined and consistent?

- It is!! *(It is guaranteed to be the case if one believes the physics behind this construction)*

To see this we have to check that the following crossing symmetry property is true:

\[
Z_{0,4}(a, b, c, d; t, p, q) = \\
\kappa \oint \frac{dz}{4\pi iz} \frac{\Gamma\left(\frac{pq}{t}z^2; p, q\right)}{\Gamma\left(z^2; p, q\right)} Z_{0,3}(a, b, z; t, p, q)Z_{0,3}(c, d, z^{-1}; t, p, q)
\]

\[
= \kappa \oint \frac{dz}{4\pi iz} \frac{\Gamma\left(\frac{pq}{t}z^2; p, q\right)}{\Gamma\left(z^2; p, q\right)} Z_{0,3}(a, c, z; t, p, q)Z_{0,3}(b, d, z^{-1}; t, p, q)
\]
This equality was proven mathematically by Fokko van de Bult (2010):

\[ \int \frac{dz}{4\pi iz} \frac{\Gamma \left( \frac{p}{t} z^\pm 2; p, q \right)}{\Gamma \left( z^\pm 2; p, q \right)} \Gamma \left( t^{\pm} a^{\pm} b^{\pm} z^{\pm}; p, q \right) \Gamma \left( t^{\pm} c^{\pm} d^{\pm} z^{\pm}; p, q \right) = \]

\[ \int \frac{dz}{4\pi iz} \frac{\Gamma \left( \frac{p}{t} z^\pm 2; p, q \right)}{\Gamma \left( z^\pm 2; p, q \right)} \Gamma \left( t^{\pm} a^{\pm} c^{\pm} z^{\pm}; p, q \right) \Gamma \left( t^{\pm} b^{\pm} d^{\pm} z^{\pm}; p, q \right). \]
“Topological” definition of $Z_{g,s}$?

- We have quite explicitly defined the functions $Z_{g,s}$ using a certain pairs-of-pants decomposition of a Riemann surface and argued that this construction is independent of the choice of such a decomposition.

- Is there a way to write an expression for $Z_{g,s}$ which will be manifestly independent of the choice of a pairs-of-pants decomposition?

- Next we will derive such an expression.
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Analytical properties of $Z_{g,s}$

- To answer the question posed on the previous slide we will study some of the analytical properties of $Z_{g,s}(\{a_i\}_{i=1}^{s}; p, q, t)$ in the parameters associated to the punctures, $a_i$.

- We consider $Z_{g,s}(a, b, c, \cdots ; p, q, t)$ (with $s > 3$) and go to a pairs-of-pants decomposition where we write this using $Z_{g,s-2}(z^{-1}, c, \cdots ; p, q, t)$ and $Z_{0,3}(a, b, z; p, q, t)$.

$$Z_{g,s}(a, b, c, \cdots ; t, p, q) =$$

$$\kappa \int \frac{dz}{4\pi iz} \frac{\Gamma \left( \frac{p+q}{2} z^{\pm 2}; p, q \right)}{\Gamma (z^{\pm 2}; p, q)} \Gamma(t^{\frac{1}{2}} a^{\pm 1} b^{\pm 1} z^{\pm 1}; p, q) Z_{g,s-2}(z^{-1}, c, \cdots ; t, p, q).$$
We can identify some of the poles in \( a \).

Varying \( a \) the location of poles in \( z \) of the integrand changes.

For special values of \( a \) pairs of poles in \( z \) pinch the integration contour and cause the integral to diverge.

Such values of \( a \) (\( |a| < 1 \)) are given by

\[
a = a_{m,n} = t^{\frac{1}{2}} p^{\frac{m}{2}} q^{\frac{n}{2}}, \quad m, n \in \mathbb{N}.
\]
Residues in \(a\)

- The residues can be easily computed since only finite number of poles in \(z\) contribute to the singularity when \(a = a_{m,n}\).

- For example, when \(a = a_{0,0} = t^{\frac{1}{2}}\) the residue is given by

\[
\text{Res}_{a \rightarrow a_{0,0}} Z_{g,s}(a, b, c, \cdots ; p, q, t) \propto Z_{g,s-1}(b, c, \cdots ; p, q, t).
\]

- That is, this residue just gives the function corresponding to the Riemann surface with one puncture less. (The proportionality factor is a simple function of \(p, q,\) and \(t\) only)

- When \(a = a_{0,1} = t^{\frac{1}{2}} q^{\frac{1}{2}}\) the residue is given by

\[
\text{Res}_{a \rightarrow a_{0,1}} Z_{g,s}(a, b, c, \cdots ; p, q, t) \propto \mathcal{S}_{(0,1)}(b) Z_{g,s-1}(b, c, \cdots ; p, q, t).
\]

where the difference operator \(\mathcal{S}_{(0,1)}(b)\) is given by

\[
\mathcal{S}_{(0,1)}(b) f(b) = \frac{\theta(\frac{t}{q} b^{-2}; p)}{\theta(b^2; p)} f(b q^{1/2}) + \frac{\theta(\frac{t}{q} b^2; p)}{\theta(b^{-2}; p)} f(b q^{-1/2}).
\]

- Up to conjugation, this operator is just the basic “Hamiltonian”, \(H_2\), of the elliptic Ruijsemaars-Schneider model.
Residues in $a$

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- When $a = a_{0,1} = t^{\frac{1}{2}} q^{\frac{1}{2}}$ the residue is given by

$$\text{Res}_{a \rightarrow a_{0,1}} Z_{g,s}(a, b, c, \cdots ; p, q, t) \propto \mathfrak{S}_{(0,1)}(b) Z_{g,s-1}(b, c, \cdots ; p, q, t).$$

where the difference operator $\mathfrak{S}_{(0,1)}(b)$ is given by

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- The residues can be easily computed since only finite number of poles in \( z \) contribute to the singularity when \( a = a_{m,n} \).

- For example, when \( a = a_{0,0} = t^{\frac{1}{2}} \) the residue is give by

\[
\text{Res}_{a \to a_{0,0}} Z_{g,s}(a, b, c, \cdots ; p, q, t) \propto Z_{g,s-1}(b, c, \cdots ; p, q, t).
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\text{Res}_{a \to a_{0,1}} Z_{g,s}(a, b, c, \cdots ; p, q, t) \propto \mathcal{G}_{(0,1)}(b) Z_{g,s-1}(b, c, \cdots ; p, q, t).
\]

where the difference operator \( \mathcal{G}_{(0,1)}(b) \) is given by

\[
\mathcal{G}_{(0,1)}(b) f(b) = \frac{\theta(t \frac{t}{q} b^{-2}; p)}{\theta(b^2; p)} f(b q^{1/2}) + \frac{\theta(t \frac{t}{q} b^2; p)}{\theta(b^{-2}; p)} f(b q^{-1/2}).
\]

- Up to conjugation, this operator us just the basic "Hamiltonian", \( \mathcal{H}_2 \), of the elliptic Ruijsenaars-Schneider model.
Properties of the difference operators

- On can compute the residues at all the poles $a = a_{m,n}$ and define corresponding difference operators $\mathcal{G}_{(m,n)}$.

- The operators $\mathcal{G}_{(m,n)}$ are self-adjoint under the “gluing” measure.

- The operator $\mathcal{G}_{(m,n)}$ commute with each other.

- The operators factorize

$$\mathcal{G}_{(m,n)} \propto \mathcal{G}_{(m,0)} \mathcal{G}_{(0,n)}.$$

- Operator $\mathcal{G}_{(0,n)}$ is obtained from $\mathcal{G}_{(n,0)}$ by exchanging $p \leftrightarrow q$.

- Operators $\mathcal{G}_{(0,n)}$ are polynomials of degree $n$ in $\mathcal{G}_{(0,1)}$.

(These properties follow from physical considerations and can be explicitly verified)
Crossing symmetry

Since our functions are invariant under crossing symmetry the difference operators satisfy a very important equality when acting on them

\[ \mathcal{G}_{(m,n)}(b) Z_{g,s}(b,c,\cdots) = \mathcal{G}_{(m,n)}(c) Z_{g,s}(b,c,\cdots). \]

The two sides of the equality correspond to two different pairs-of-pants decompositions.
For example we can act with $\mathcal{G}_{(0,1)}$ on $Z_{0,3}(a, b, c; p, q, t)$,

$$\mathcal{G}_{(0,1)}(a)Z_{0,3}(a, b, c; p, q, t) \propto \Gamma \left( \sqrt{\frac{t}{q}} a^{\pm 1} b^{\pm 1} c^{\pm 1}; p, q \right) \times$$

$$\left[ \frac{\theta(\frac{t}{q} a^{-2}; p) \theta(\sqrt{\frac{t}{q}} a b^{\pm 1} c^{\pm 1}; p)}{\theta(a^2; p)} + \frac{\theta(\frac{t}{q} a^2; p) \theta(\sqrt{t} a^{-1} b^{\pm 1} c^{\pm 1}; p)}{\theta(a^{-2}; p)} \right].$$

One can check that the combination of theta functions on the second line is invariant under permutations of $a$, $b$, and $c$. 

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Topological expression for $Z_{g,s}$

Defining the eigenfunctions of the difference operators by $\psi^\lambda$ and also defining the eigenvalues as

$$\mathcal{G}_{(1,0)}(a) \cdot \psi^\lambda(a; p, q, t) = E_\lambda(p, q, t) \psi^\lambda(a; p, q, t),$$

we (at least formally) expand the functions in $\psi^\lambda$ and obtain (for brevity $p, q, t$ are dropped here)

$$\mathcal{G}_{(1,0)} Z_{0,3} = \sum_{\alpha, \beta, \gamma} C_{\alpha \beta \gamma} E_\alpha \psi^\alpha(a) \psi^\beta(b) \psi^\gamma(c) = \sum_{\alpha, \beta, \gamma} C_{\alpha \beta \gamma} E_\beta \psi^\alpha(a) \psi^\beta(b) \psi^\gamma(c)$$

$$= \sum_{\alpha, \beta, \gamma} C_{\alpha \beta \gamma} E_\gamma \psi^\alpha(a) \psi^\beta(b) \psi^\gamma(c).$$

This implies that the functions are diagonal in the basis of $\psi^\alpha$ (assuming the spectrum is not degenerate)

$$Z_{0,3} = \sum_{\alpha} C_{\alpha} \psi^\alpha(a) \psi^\alpha(b) \psi^\alpha(c) = \Gamma \left( t^{\frac{1}{2}} a^{\pm 1} b^{\pm 1} c^{\pm 1}; p, q \right)$$
Topological expression for $Z_{g,s}$ II

- By using the fact that the residue at $a = t^{1/2}$ removes a puncture the structure constants $C_\alpha$ are also fixed,

\[ C^{-1}_\alpha \propto \text{Res}_{a \to t^{1/2}} \psi^\alpha (a). \]

- Finally the function corresponding to a generic Riemann surface can be written as

\[ Z_{g,s}(\{a_i\}_{i=1}^s; p, q, t) = \sum_\lambda C^2_{g-2+s} \prod_{\ell=1}^s \psi^\lambda (a_\ell). \]

- This result can be explicitly checked against the integral representations of the functions at-least in some limits of the parameters. E.g., Macdonald limit: $p = 0$ (or $q = 0$), Schur limit $q = t$ (or $p = t$). In the latter limit the dependence on $p$ ($q$) drops out.

- At the full elliptic level this gives a concrete relation between the eigenfunctions of the elliptic RS model and the integral representations of our functions.
We have defined a set of functions corresponding to Riemann surfaces, $Z_{g,s}(\{a_i\}; p, q, t)$.

These functions depend on $A_1$ parameters, $a_i$, corresponding to each puncture of the surface, as well as on three additional parameters $p, q, t$.

$A_1$ symmetric $\rightarrow Z_{g,s}(\{a_i\}; p, q, t)$ invariant under $a_i \rightarrow 1/a_i$.

Two ways to write the expressions for the functions: first, as contour integrals of elliptic Gamma functions, and second in terms of eigen-functions of elliptic RS models.

All this can be generalized to $A_{N-1}$, though the generalization is quite non-trivial.
Intermediate summary

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- $A_1$ symmetric $\rightarrow Z_{g,s}(\{a_i\}; p, q, t)$ invariant under $a_i \rightarrow 1/a_i$.

- Two ways to write the expressions for the functions: first, as contour integrals of elliptic Gamma functions, and second in terms of eigen-functions of elliptic RS models.

- All this can be generalized to $A_{N-1}$, though the generalization is quite non-trivial.
$A_{N-1}$ symmetric Kingdom
Similarly to the $A_1$ case we associate functions to punctured Riemann surfaces.

However, now we have different species of punctured classified by partitions of $N$. Here is an example of $A_{25}$ puncture,

The parameters are constrained to satisfy $\left(ab\right)^5\left(cde\right)^4f^2gh = 1$. These can be thought of as parametrizing the group $S(U(2)U(3)U(1)U(2))$.

For simplicity, we will discuss here only $SU(N)$ and $U(1)$ punctures. We will denote our functions by $Z_{g,(s,n,...)}^{(N)}$ where $s$ counts $SU(N)$ and $n$ $U(1)$ punctures.
Basic example

The function corresponding to two $SU(N)$ punctures (single row) and one $U(1)$ puncture (one column with $N-1$ boxes and another one with a single box) is given by a product of elliptic Gamma functions,

\[ Z_{0,(2,1,0,0,\ldots)}^{(N)}(a, \{b_i\}_{i=1}^N, \{c_i\}_{i=1}^N; t, p, q) = \prod_{i,j=1}^{N} \Gamma \left( t^{1/2} (a b_i c_j)^{\pm 1} ; p, q \right). \]

Here \( \prod_{i=1}^{N} b_i = \prod_{i=1}^{N} c_i = 1. \)
Gluing

Given the functions corresponding to two Riemann surfaces, $C_{g_1,(s_1,\ldots)}$ and $C_{g_2,(s_2,\ldots)}$, one can obtain the function corresponding to $C_{g_1+g_2,(s_1+s_2-2,\ldots)}$ by “gluing” the two surfaces along an $SU(N)$ puncture.

$$Z_{g_1+g_2,(s_1+s_2-2,\ldots)}(a_1,\ldots,a_{s_1-1},b_1,\ldots,b_{s_2-1},\ldots; t, p, q) =$$

$$\frac{\kappa^{N-1}}{N!} \oint \frac{dz_\ell}{2\pi iz_\ell} \prod_{\ell=1}^{N-1} \prod_{i \neq j}^{N} \frac{\Gamma \left( \frac{p q}{t} \frac{z_i}{z_j}; p, q \right)}{\Gamma \left( \frac{z_i}{z_j}; p, q \right)} Z_{g_1,(s_1,\ldots)}(a_1,\ldots,a_{s_1-1},z,\ldots; t, p, q) \times Z_{g_2,(s_2,\ldots)}(b_1,\ldots,b_{s_2-1},z^{-1},\ldots; t, p, q).$$
As with the $A_1$ case, we construct the functions for generic Riemann surfaces here by making a pairs-of-pants decomposition and then gluing together three-punctured spheres.

Unlike the $A_1$ case, here we have many different three-punctured spheres determined by the types of the three punctures.

In particular there are many different consistency checks we have to perform: all the four-punctured spheres should be crossing symmetry invariant.

For example:
Gluing together two three-punctured spheres with two $SU(N)$ punctures and a $U(1)$ puncture we defined above, we obtain an $A_{N-1}$ generalization of the identity for $A_1$:

\[ \oint \prod_{i=1}^{N-1} \frac{dz_i}{2\pi iz_i} \prod_{i \neq j}^{N} \frac{\Gamma \left( \frac{p}{t} z_i/z_j; p, q \right)}{\Gamma \left( z_i/z_j; p, q \right)} \prod_{i,j=1}^{N} \Gamma \left( t^{\frac{1}{2}} (ab_i z_j)^{\pm 1}; p, q \right) \Gamma \left( t^{\frac{1}{2}} (cd_i z_j^{-1})^{\pm 1}; p, q \right) = \]

\[ \oint \prod_{i=1}^{N-1} \frac{dz_i}{2\pi iz_i} \prod_{i \neq j}^{N} \frac{\Gamma \left( \frac{p}{t} z_i/z_j; p, q \right)}{\Gamma \left( z_i/z_j; p, q \right)} \prod_{i,j=1}^{N} \Gamma \left( t^{\frac{1}{2}} (cb_i z_j)^{\pm 1}; p, q \right) \Gamma \left( t^{\frac{1}{2}} (ad_i z_j^{-1})^{\pm 1}; p, q \right). \]

Checked this in expansion in the parameters.
I have not told you however what the functions associated to the general three-punctured spheres are:

This is the main physically interesting question we want to answer!

Physics-wise general three-punctured spheres correspond to complicated (strongly-coupled) objects.
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Physics-wise general three-punctured spheres correspond to complicated (strongly-coupled) objects.
More constraints on the functions

- To answer the question posed on the previous slide one has to provide more information about the functions.

- For general $A_{N-1}$ such information is provided by specifying some relations between functions associated to Riemann surfaces with different numbers of punctures.

- For example, in the $A_2$ case the single relation sufficient to fix all the functions is

$$Z_{0,(2,2)}(a, b, z, y; p, q, t) = \kappa \oint \frac{du}{2\pi i u} \frac{\Gamma \left( \frac{pq}{t} u^{\pm 2}; p, q \right) \Gamma \left( t^{\frac{1}{2}} u^{\pm 1} \left( \frac{a}{b} \right)^{\pm \frac{3}{2}}; p, q \right)}{\Gamma (u^{\pm 2}; p, q) Z_{0,(3,0)} \left( \left\{ \frac{\sqrt{ab}}{u}, \sqrt{abu}, \frac{1}{ab} \right\}, z, y; p, q, t \right)$$

- This constraint can be actually solved!! (Spiridonov, Warnaar - 2004). (The left-hand-side can be obtained by gluing two $Z_{0,(2,1)}$.) One can obtain explicit contour integral expression for $Z_{0,(3,0)}$ and check that all the crossing symmetries are satisfied and thus the construction of the $A_2$ functions is consistent.

- $Z_{0,(3,0)}$ has three $SU(3)$ factors but the symmetry is actually enhanced to $E_6$.

- Similar constraints can be written down systematically for higher rank cases.
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More constraints on the functions

- To answer the question posed on the previous slide one has to provide more information about the functions.

- For general $A_{N-1}$ such information is provided by specifying some relations between functions associated to Riemann surfaces with different numbers of punctures.

- For example, in the $A_2$ case the single relation sufficient to fix all the functions is

$$Z_{0,(2,2)}(a, b, z, y; p, q, t) =$$

$$\kappa \oint \frac{du}{2\pi i u} \frac{\Gamma\left(\frac{pq}{t} u^{\pm 2}; p, q\right) \Gamma\left(t^{\frac{1}{2}} u^{\pm 1} \left(\frac{a}{b}\right)^{\pm \frac{3}{2}}; p, q\right)}{\Gamma(u^{\pm 2}; p, q)} Z_{0,(3,0)}\left(\left\{\frac{\sqrt{ab}}{u}, \sqrt{abu}, \frac{1}{ab}\right\}, z, y; p, q, t\right)$$

- This constraint can be actually solved!! (Spiridonov, Warnaar - 2004). (The left-hand-side can be obtained by gluing two $Z_{0,(2,1)}$.) One can obtain explicit contour integral expression for $Z_{0,(3,0)}$ and check that all the crossing symmetries are satisfied and thus the construction of the $A_2$ functions is consistent.

- $Z_{0,(3,0)}$ has three $SU(3)$ factors but the symmetry is actually enhanced to $E_6$.

- Similar constraints can be written down systematically for higher rank cases.
Some spheres with exceptional symmetries

- $A_2$ three-punctured sphere with $E_6$ symmetry

- $A_3$ three-punctured sphere with $E_7$ symmetry, and $A_5$ three-punctured sphere with $E_8$ symmetry
Topological expressions for the $A_{N-1}$ case

- As we did for $A_1$ we can seek for a more topological description of the functions $Z_{g,(s,n,...)}$.

- One can generalize in a straightforward way the poles/residues analysis we have done there.

- Consider the following pairs-of-pants decomposition of a generic Riemann surface

$$Z_{g,(s,n,...)}(a, b, c, \cdots ; t, p, q) = \frac{\kappa^{N-1}}{N!} \oint \prod_{i=1}^{N-1} \frac{dz_i}{2\pi iz_i} \prod_{i \neq j} \frac{\Gamma\left(\frac{p q z_i}{t z_j}; p, q\right)}{\Gamma\left(\frac{z_i}{z_j}; p, q\right)} \times \prod_{i,j=1}^{N} \Gamma\left(t^{\frac{1}{2}}(a b_i z_j)^{\pm 1}; p, q\right) Z_{g,(s,n-1,...)}(z^{-1}, c, \cdots ; t, p, q).$$
Poles and residues in \( a \)

- We look for pole in \( a \). A class of such poles is located at
  \[
a = a_{m,n} = t^{\frac{1}{2}} p^\frac{m}{N} q^\frac{n}{N}, \quad m, n \in \mathbb{N}.
\]

- The residues again are easily computed. For example, the residue at \( a_{0,1} \) is given by
  \[
  \text{Res}_{a \rightarrow a_{0,1}} Z_{g,(s,n,...)}(a, b, c, \cdots ; p, q, t) \propto \mathcal{G}_{(0,1)}(b) \ Z_{g,(s,n-1,...)}(b, c, \cdots ; p, q, t).
\]

  where the difference operator \( \mathcal{G}_{(0,1)}(b) \) is given by

  \[
  \mathcal{G}_{(0,1)}(b) f(b) = \left( \prod_{i \neq j} \Gamma \left( tb_i/b_j; p, q \right) \right) \mathcal{H}_2(b) \left( \prod_{i \neq j} \Gamma \left( tb_i/b_j; p, q \right) \right)^{-1} f(b).
\]

- Here \( \mathcal{H}_2(b) \) is the basic “Hamiltonian” of the elliptic RS model.
Topological expressions

- Exploiting crossing symmetry and all the constraints, after the dust settles, we can write the following expressions for the functions.

- The functions corresponding to Riemann surfaces with only $SU(N)$ punctures are given by

$$Z_{g,(s,0,...)} = \sum_{\lambda} \prod_{\ell=1}^{s} \left( \prod_{i \neq j}^{N} \Gamma(t b_i^{(\ell)}/b_j^{(\ell)}; p, q) \right) \phi_{\lambda}(b^{(\ell)}; p, q, t) \phi_{\lambda}(t^{1-N}/2, \ldots, t^{N-1}/2; p, q, t)^{2g-2+s}.$$  

  Here $\phi_{\lambda}$ are eigenfunctions of the elliptic RS model and $\left( \prod_{i \neq j}^{N} \Gamma(t b_i/b_j; p, q) \right) \phi_{\lambda} = \psi_{\lambda}$ are orthonormal eigenfunctions of $\mathcal{G}_{(m,n)}$. We can explicitly check in degeneration limits where the eigenfunctions are explicitly known that the above agrees with other, integral, representations of the functions.

- In case we have one $U(1)$ puncture and two $SU(N)$ punctures the function is given by a product of $2N^2$ elliptic Gamma functions and has the following “topological” expression:

$$Z_{0,(2,1,...)} = \prod_{i \neq j}^{N} \Gamma(t b_i^{(\ell)}/b_j^{(\ell)}; p, q) \prod_{\ell=1}^{2} \left( \prod_{i \neq j}^{N} \Gamma(t b_i^{(\ell)}/b_j^{(\ell)}; p, q) \right) \times$$

$$\sum_{\lambda} \prod_{\ell=1}^{2} \phi_{\lambda}(b^{(\ell)}; p, q, t) \phi_{\lambda}(t^{2-N}/2, \ldots, t^{N-2}/2; a, a^{1-N}; p, q, t).$$

- Such expressions can be systematically written for functions corresponding to generic Riemann surfaces with generic punctures.
$A_1$ non-symmetric Empire
The $A_1$ construction can be generalized in yet another way.

We introduce an integer positive parameter $r$. The case of $r = 1$ is the one we discussed so far.

Each puncture on the Riemann surface is labeled now by an $SU(2)$ parameter $a_i$ and an integer $m_i$ defined mod $r$. We are looking for functions associated to Riemann surfaces with this data, $Z_{g,s}(\{a_\ell, m_\ell\}_{\ell=1}^s; p, q, t)$. The functions are not symmetric for general $m_i$.

A starting point is the function corresponding to a three-punctured sphere:

$$Z_{0,3}(\{a_\ell, m_\ell\}_{\ell=1}^s; p, q, t) = \left(\frac{pq}{t}\right)^{\frac{1}{4}} \sum_{s_\ell = \pm 1} (\sum_{\ell=1}^3 s_\ell m_\ell)r - \frac{([\sum_{\ell=1}^3 s_\ell m_\ell]r)^2}{r^2} \prod_{s_\ell = \pm 1} \Gamma(t^{\frac{1}{2}} p^{[\sum_{\ell=1}^3 s_\ell m_\ell]r} \prod_{\ell=1}^3 a_\ell^{s_\ell}; pq, p') \Gamma(t^{\frac{1}{2}} q^{r-[\sum_{\ell=1}^3 s_\ell m_\ell]r} \prod_{\ell=1}^3 a_\ell^{s_\ell}; pq, q')$$
The gluing is also modified. Given the functions corresponding to two Riemann surfaces, \( C_{g_1, s_1} \) and \( C_{g_2, s_2} \), one can obtain the function corresponding to \( C_{g_1+g_2, s_1+s_2-2} \) as before

\[
Z_{g_1+g_2,s_1+s_2-2}(\{a_i, m^a_i\}_{i=1}^{s_1-1}, \{b_i, m^b_i\}_{i=1}^{s_2-1}; t, p, q) \propto \\
\sum_{n=0}^{\lfloor r/2 \rfloor} l_0^V (p, q, t, n) \int \frac{dz}{4\pi iz} \frac{\Gamma(\frac{pq}{t} p^+[\pm 2n]_r z^{\pm 2}; pq, p^r) \Gamma(\frac{pq}{t} q^r-[\pm 2n]_r z^{\pm 2}; pq, q^r)}{\Gamma(\frac{pq}{t} p^-[\pm 2n]_r z^{\pm 2}; pq, p^r) \Gamma(q^r-[\pm 2n]_r z^{\pm 2}; pq, q^r)} \times \\
Z_{g_1,s_1}(\{a_i, m^a_i\}_{i=1}^{s_1-1}, \{z, n\}; t, p, q)Z_{g_2,s_2}(\{b_i, m^b_i\}_{i=1}^{s_2-1}, \{z^{-1}, [-n]_r\}; t, p, q).
\]

The crossing symmetry can be checked to hold (it was done in some limits).
Difference operators

- We can repeat again the analysis of poles and residues.

- The residues are given again by difference operators. However, now they take the schematic form

\[ \text{Res}_{a \rightarrow a^*} Z_{g,s}(\{a, 0\}, \{b, m\}, \cdots) \sim (\mathcal{O}^n_m)_{a^*} (b, m) Z_{g,s-1}(\{b, n\}, \cdots). \]

In general all the components of the matrix \((\mathcal{O}^n_m)_{a^*}\) are non-zero.

- In some limits however these difference operators simplify. One such limit is taking \(p \rightarrow 0\) (Macdonald). Here the difference operators are proportional to \(\delta^n_m\). For example the operator computing the basic non-trivial residue is schematically given by

\[ (\mathcal{O}^n_m)_{a^* = t^{\frac{1}{2}} q^{\frac{1}{2}}} \sim K (Y_1 + Y_2) K^{-1}. \]

Here \(K\) is a simple product of elliptic Gamma functions and \(Y_i\)s are \(A_1\) Cherednik difference operators.

- The eigenfunctions here are given in terms of non-symmetric Macdonald polynomials and our functions are naturally expressible in terms of these.

- All these has a “straightforward” generalization to \(A_{N-1}\) case.
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\[ \text{Res}_{a \to a^*} Z_{g,s}(\{a,0\}, \{b,m\}, \cdots) \sim (\mathcal{D}^n_m)_{a^*} (b,m) Z_{g,s-1}(\{b,n\}, \cdots). \]

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Comments
The functions $Z_{g,(s,n,...)}$ in the symmetric “kingdom” are superconformal indices, aka twisted supersymmetric partition function on $S^3 \times S^1$, of theories of class $S$ labeled by the corresponding Riemann surface.

The functions $Z_{g,(s,n,...)}$ in the non symmetric “empire” are lens space indices, aka twisted supersymmetric partition function on $S^3/Z_r \times S^1$, of theories of class $S$ labeled by the corresponding Riemann surface.

Let us stress again that most of the theories in class $S$ are strongly-coupled meaning that a priori direct computations for them are not possible. However, by exploiting dualities (= extra constraints on the functions) and certain RG flows (= the residue calculus) one can fix their indices.

The indices have physical meaning and thus have to be consistent with what we expect from the theories on physical grounds.

And they are. (Eg symmetry enhancements, spectrum of protected operators, constraints, dualities,...)
Some References

The index of theories of class $S$ was discussed in a series of papers:

- $A_1$ - Gadde, Pomoni, Rastelli, SR (2009)
- $A_2$ - Gadde, Rastelli, SR, Yan (2010)
- $A_{N-1}$ in Macdonald limit - Gadde, Rastelli, SR, Yan (2011x2)
- $A_{N-1}$ and difference operators - Gaiotto, Rastelli, SR (2012)
- $\mathcal{N} = 2$ lens index (definition) - Benini, Nishioka, Yamazaki (2011)
- $A_1$ lens index in “Schur” limit - Alday, Bullimore, Fluder - (2013)
- $A_{N-1}$ lens index and difference operators - SR, Yamazaki (2013)
- More on $A_{N-1}$ index and exceptional symmetries - Gaiotto, SR (2012)
- $D_n$ index - Lemos, Peelaers, Rastelli (2013); Mekareeya, Song, Tachikawa (2012)
- Index and exceptional instantons - Hanany, Mekareeya, SR (2012); Keller, Song (2012)
- More related topics - Spiridonov, Vartanov (2010); Nishioka, Tachikawa, Yamazaki (2011); Tachikawa (2012); SR (2012); Beem, Gadde (2012); Gadde, Maruyoshi, Tachikawa, Yan (2013); Maruyoshi, Tachikawa, Yan, Yonekura (2013); Gadde, Gukov (2013);

...
Hopefully there will be fruitful interactions between the more QFT oriented and the more Math oriented communities.

Thank You!!