Universal computation by multi-particle quantum walk

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Quantum analogue of classical random walk

Two types:
Discrete-time and Continuous-time
Quantum analogue of classical random walk

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Framework for developing algorithms

Glued trees traversal [Childs et. al 03]
NAND tree [FGG08]
Element Distinctness [Ambainis 03]

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Tunneling of photons between adjacent optical waveguides [Perets et. al 08]
Single atom moving in a 1D optical lattice [Karski et. al. 09]
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Quantum Walk

Testbed for Experiments

Tunneling of photons between adjacent optical waveguides [Perets et. al 08]
Single atom moving in a 1D optical lattice [Karski et. al. 09]

Computational model

Universality [Childs 09]
The quantum walks we consider here are in **continuous time**: generated by a time-independent Hamiltonian associated with a graph.
**Theorem:** An n-qubit, g-gate quantum circuit can be efficiently simulated by a multi-particle quantum walk of n+1 particles on a graph with poly(n,g) vertices.
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- Establishes the computational power of interacting many-body systems such as the Bose-Hubbard model, fermions with nearest neighbour interactions, and more.
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Our method for performing universal computation exploits the connection between quantum walk and a discrete version of scattering theory [Farhi Gutmann 98]
Single Particle Quantum Walk
- Definition and example
- Analysis using scattering theory
- How to use it to perform a computation [Childs 09]

Multi-Particle Quantum Walk
- Definition and examples
- Our universality construction
Single-particle quantum walk on a graph $G$
Single-particle quantum walk on a graph G

The quantum walker is a particle that can be located at any vertex in the graph.
Single-particle quantum walk on a graph $G$

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The quantum walk is generated by the time-independent Hamiltonian

$$H_G = \sum_{(i,j) \in E(G)} |i\rangle\langle j| + |j\rangle\langle i|$$
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$$H_G = \sum_{(i,j) \in E(G)} |i\rangle \langle j| + |j\rangle \langle i|$$

$$|\psi(t)\rangle = e^{-iH_G t}|\psi(0)\rangle$$
Example: quantum walk on an infinite path

\[ H_G = \sum_{x = -\infty}^{\infty} (|x\rangle\langle x + 1| + |x + 1\rangle\langle x|) \]
Example: quantum walk on an infinite path

\[ \ldots \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad \ldots \]

\[ H_G = \sum_{x=-\infty}^{\infty} (|x\rangle\langle x+1| + |x+1\rangle\langle x|) \]

The eigenstates are momentum states

\[ |\tilde{k}\rangle = \sum_{x=-\infty}^{\infty} e^{-ikx} |x\rangle, \quad k \in [-\pi, \pi] \]

\[ H_G |\tilde{k}\rangle = 2 \cos k |\tilde{k}\rangle \]
Example: quantum walk on an infinite path (continued)

Wave packet
A normalized state with most of its amplitude on momentum states near a particular value $k$.

For example,

$$|\chi_{L,k}\rangle = \frac{1}{\sqrt{L}} \sum_{x=1}^{L} e^{-ikx} |x\rangle$$

Has support on $L$ vertices
Example: quantum walk on an infinite path (continued)

Wave packet
A normalized state with most of its amplitude on momentum states near a particular value $k$.

e.g.,

$$|\chi_{L,k}\rangle = \frac{1}{\sqrt{L}} \sum_{x=1}^{L} e^{-ikx} |x\rangle$$

Has support on $L$ vertices

Propagates with speed

$$\left| \frac{dE}{dk} \right| = |2 \sin k|$$
Example: quantum walk on an infinite path (continued)

Wave packet propagation
Quantum walk and scattering theory

A finite graph $\hat{G}$ attached to two semi-infinite paths.
Quantum walk and scattering theory

A finite graph $\hat{G}$ attached to two semi-infinite paths.

What happens if we start at $t=0$ in a wave packet prepared on the left with incoming momentum $k$ (moving towards the finite graph)?
Quantum walk and scattering theory

After scattering there is a reflected wave packet and a transmitted wave packet.

\[ T(k) \quad \hat{G} \quad R(k) \]
Quantum walk and scattering theory

After scattering there is a reflected wave packet and a transmitted wave packet.

\[ T(k) \quad + \quad R(k) \]

\[ \hat{G} \]

\[ k \rightarrow \]

\[ k \leftarrow \]
Quantum walk and scattering theory

For scattering from the right there are also (different) transmission and reflection coefficients.

**S-matrix:**

2x2 unitary matrix containing the 4 coefficients that describe scattering.
Quantum walk and scattering theory

More generally,
Quantum walk and scattering theory

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Here the $S$-matrix is an $N \times N$ unitary matrix.
Quantum walk and scattering theory

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Here the $S$-matrix is an $N \times N$ unitary matrix.

Incoming wave packets map to outgoing wave packets with amplitudes determined by the $S$-matrix. [Farhi Gutmann 98] [Childs 09][Varbanov, Brun 09][Childs, Gosset 12].
How to use quantum walk to perform a **one-qubit** unitary [Childs 09]

Encode a qubit in the location of the particle with momentum $\frac{\pi}{4}$

Encode $\ket{0}$

Encode $\ket{1}$
How to use quantum walk to perform a **one-qubit** unitary [Childs 09]

Connect an obstacle (a subgraph) to two input paths and two output paths.

\[ \hat{G}_U \]

\( \hat{G}_U \) is designed so that the scattering process implements U on the encoded qubit.
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\(\hat{G}_U\) is designed so that the scattering process implements \(U\) on the encoded qubit.

i.e., its S-matrix at momentum \(\pi/4\) is equal to

\[
S = \begin{pmatrix}
0 & U' \\
U & 0
\end{pmatrix}
\]

Each block is 2 x 2
How to use quantum walk to perform a one-qubit unitary [Childs 09]

Finite graphs $\hat{G}_U$ that implement one-qubit gates:

To obtain a graph implementing two of these unitaries in series, concatenate the two graphs.

$$U_{\text{phase}} = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & 1 \end{pmatrix}$$

$$U_{\text{basis}} = -\frac{i}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$
This can be generalized to simulate an n-qubit quantum circuit by single-particle quantum walk [Childs 2009].

The graph for the quantum walk grows exponentially with n.
Single Particle Quantum Walk
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Multi-particle quantum walk
Multi-particle quantum walk

For $m$ distinguishable particles on a graph $G$, the Hilbert space is spanned by

$$\{|i_1, \ldots, i_m\rangle: i_1, \ldots, i_m \in V(G)\}$$
Multi-particle quantum walk

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We can also consider **indistinguishable particles**
Multi-particle quantum walk

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We can also consider **indistinguishable particles**

Bosons live in the symmetric subspace

Fermions live in the antisymmetric subspace
Hamiltonian for a multi-particle quantum walk of $m$ particles

$$H_{G}^{(m)} = \sum_{w=1}^{m} \sum_{i,j \in E(G)} \left( |i\rangle\langle j|_{w} + |j\rangle\langle i|_{w} \right) + U$$

Independent single-particle quantum walks for each particle

Interaction between particles
Hamiltonian for a multi-particle quantum walk of $m$ particles

$$H_{G}^{(m)} = \sum_{w=1}^{m} \sum_{i,j \in E(G)} \left( |i\rangle \langle j|_{w} + |j\rangle \langle i|_{w} \right) + \mathcal{U}$$

- Independent single-particle quantum walks for each particle
- Interaction between particles

Our results apply to a broad class of local interaction types including

**nearest neighbor interactions:** e.g.,

$$\mathcal{U} = J \sum_{(i,j) \in E(G)} \hat{n}_i \hat{n}_j$$

number operator that counts the number of particles at vertex $i$
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Interaction between particles

Our results apply to a broad class of local interaction types including

**nearest neighbor interactions:** e.g., $$U = J \sum_{(i,j) \in E(G)} \hat{n}_{i}\hat{n}_{j}$$

**on-site interactions,** e.g., $$U = J \sum_{i \in V(G)} \hat{n}_{i} (\hat{n}_{i} - 1)$$

number operator that counts the number of particles at vertex $i$
Universal computation by multi-particle quantum walk

I will now describe how a quantum computation can be efficiently simulated by a multi-particle quantum walk of indistinguishable particles on a graph.
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Main idea:

Encode each qubit using a particle that moves as a wave packet
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Single-qubit gates: The particles scatter from a sequence of subgraphs while remaining far apart.
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I will now describe how a quantum computation can be efficiently simulated by a multi-particle quantum walk of indistinguishable particles on a graph.

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Encode each qubit using a particle that moves as a wave packet

Single-qubit gates: The particles scatter from a sequence of subgraphs while remaining far apart.

Two qubit gate: Two particles are routed towards one another. The interaction between the particles implements the gate, and then the particles move away from one another.
Step 1: Rewrite the circuit

Circuit is a given as a sequence of single qubit gates and CP gates

\[
\text{CP} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -i
\end{pmatrix}
\]
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Circuit is a given as a sequence of single qubit gates and CP gates

Rewrite the circuit using the circuit identity

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“mediator” qubit
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\end{pmatrix} \]

The circuit now has \( n \) “computational qubits”, one “mediator qubit” and
- Single qubit gates on the computational qubits
- Hadamard gates on the mediator qubit
- CP gates between the mediator and a computational qubit
Encode the n+1 qubits using n+1 particles

Computational qubit $\rightarrow$ wave packet with momentum $\frac{\pi}{4}$

Mediator qubit $\rightarrow$ wave packet with momentum $\frac{\pi}{2}$
One-qubit gates

Graphs from [Childs 09] implement one-qubit gates at momentum $\pi/4$

Graph from [Blumer et. al. 11] that implements a Hadamard gate at momentum $\pi/2$
CP gate between the mediator and a computational qubit
CP gate between the mediator and a computational qubit

To implement the CP gate we use a simple fact about two particle scattering.
Aside: Two particle scattering for indistinguishable particles

\[ t=0 \]
Aside: Two particle scattering for indistinguishable particles

For indistinguishable particles, there is only one possible final state:

\[ e^{i\theta} \]
Aside: Two particle scattering for indistinguishable particles

For indistinguishable particles, there is only one possible final state:

\[ e^{i\theta} \]

The phase is determined by the details of the interaction.
CP gate between the mediator qubit and a computational qubit

We design a graph that routes the two particles onto a long path of vertices (moving towards each other) only if both qubits are in the encoded logical state 1. This implements

\[ C\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{pmatrix} \]
CP gate between the mediator qubit and a computational qubit

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0 & 0 & 1 & 0 \\
0 & 0 & 0 & e^{i\theta}
\end{pmatrix}
\]

Generically we can choose an integer a such that

\[e^{ia\theta} \approx -i\]

and so repeating the \(C\theta\) gate a times we approximate the CP gate.
Cθ gate between the mediator qubit and a computational qubit

“Momentum switch graph”:
- A particle with momentum $\frac{\pi}{4}$ follows the single line.
- A particle with momentum $\frac{\pi}{2}$ follows the double line.
Cθ gate between the mediator qubit and a computational qubit

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See also follow-up work on momentum switches [Childs, Gosset, Nagaj, Raha, Webb 14]
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Graph that implements the Cθ gate:
\( C_{\theta} \) gate between the mediator qubit and a computational qubit

\[ \begin{align*}
1 & \quad 2 = 1 \quad 2 \\
3 & \quad 3
\end{align*} \]

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Graph that implements the \( C_{\theta} \) gate:

\[ \begin{align*}
0_{c,\text{in}} & \rightarrow 0_{c,\text{out}} \\
1_{c,\text{in}} & \rightarrow 1_{c,\text{out}} \\
0_{m,\text{in}} & \rightarrow 0_{m,\text{out}} \\
1_{m,\text{in}} & \rightarrow 1_{m,\text{out}}
\end{align*} \]

Case |00>
C\(\theta\) gate between the mediator qubit and a computational qubit

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Graph that implements the C\(\theta\) gate:

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Graph that implements the $C\theta$ gate:

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“Momentum switch graph”:

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Graph that implements the Cθ gate:

\[
\begin{align*}
0_{c,\text{in}} & \quad 1_{c,\text{in}} \quad 0_{c,\text{out}} \\
0_{m,\text{in}} & \quad 1_{m,\text{in}} \quad 0_{m,\text{out}}
\end{align*}
\]
Example
Error bound

Each particle is initialized at $t=0$ as a wave packet of length $L$.
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For a $g$-gate, $n$-qubit circuit, the graph has $O(ngL)$ vertices.

The total evolution time is $O(gL)$. 
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The total evolution time is $O(gL)$.

We prove that by taking $L = \text{poly}(n,g)$ the error in the simulation can be made arbitrarily small.

E.g., for the Bose-Hubbard model $L = O(n^{12}g^4)$. 
Refinements and extensions

The scheme as described here uses \( n+1 \) indistinguishable particles on a nonplanar graph of maximum degree 5 to simulate an \( n \)-qubit quantum circuit.
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Variations:

• Planar graphs of maximum degree 4

• Distinguishable particles with nearest neighbor interactions

• Quasiparticles: e.g., magnons in the (time-independent) Heisenberg model
Refinements and extensions

The scheme as described here uses \( n + 1 \) indistinguishable particles on a nonplanar graph of maximum degree 5 to simulate an \( n \)-qubit quantum circuit.

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- Distinguishable particles with nearest neighbor interactions
- Quasiparticles: e.g., magnons in the (time-independent) Heisenberg model

Follow up work:

- Particles with internal degrees of freedom [Bao, Hayden, Salton, Thomas 2014]
- Better error bounds if one allows the interactions to vary spatially within the graph [Thompson, Gokler, Lloyd, Shor 2015]
Open problems

Can the initial state in our scheme be simplified (e.g., start each particle at a single vertex)?
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Algorithms based on multi-particle quantum walk?